Turbulence - Methods and Applications, 2025-6

Booklet of articles to skim through before our five "Bilingualism" classes online

N.B. Location of Penelope Moffatt's virtual classroom:
https://univ-poitiers.webex.com/meet/penelope.moffatt

Here is an optimistic table of what we will attempt to read and watch (and maybe even partially understand), before when. Each time, think of at least one question that you would ask the author of the article, if you were to meet them. (To allow some space for creativity.)

For when ***What***	*Article(s) to read BEFORE THE CLASS or at least skim through (read "diagonally")*	*Anything else to do? (eg. listening)*
Tuesday 23rd September at 4.00pm **Reynolds**	Reynolds 1895: *On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion*	Read this historic article in several sittings – it is tough, but illuminating. Be ready to summarize part 1 **or** part 2 to a classmate. Watch the rest of Steve Brunton's Introduction to Turbulence and do the questions on it, on the *Turbulence Bilingue* site.
Tues. 21st October at 4.00pm **Kolmogorov and more**	Oboukov 1961: *Some specific features of atmospheric turbulence* Kolmogorov 1961: *A refinement of previous hypotheses…* in French AND in English	Watch Steve Brunton's 33 minute talk on *Turbulence Closure Models: RANS and LES* and take some notes. Find a few terms that surprise you in the French version of Kolmogorov's paper, and how they have been translated into English.
Tues. 19th November at 4.00pm **Homogeneous turbulence**	H.K. Moffatt, 2012 *Homogeneous turbulence: an introductory review*	Be ready to summarize one section of the article, orally.
Tues. 3rd December at 4.00pm **Popular fallacies about turbulence**	Peter Bradshaw, 1994: *Turbulence – the chief outstanding difficulty of our subject*	Be ready to summarise one section of the article. (We will decide who summarizes which section during the 19th Nov. class.)
Tues 7th January 2025 at 4.00pm **Stars**	Canuto et al, 1998: *Turbulence in Astrophysics: Stars* (tough but fascinating)	**Poster check:** check with me on any content that you are writing in English of which you are unsure.

NB. Final poster session: Tuesday 13th January 2026 from 2pm to 4pm.

62.

ON THE DYNAMICAL THEORY OF INCOMPRESSIBLE VISCOUS FLUIDS AND THE DETERMINATION OF THE CRITERION.

[*From the* "Philosophical Transactions of the Royal Society," 1895.]

(*Read* May 24, 1894.)

SECTION I.

Introduction.

1. THE equations of motion of viscous fluid (obtained by grafting on certain terms to the abstract equations of the Eulerian form, so as to adapt these equations to the case of fluids subject to stresses depending in some hypothetical manner on the rates of distortion, which equations Navier* seems to have first introduced in 1822, and which were much studied by Cauchy† and Poisson‡) were finally shown by St Venant§ and Sir Gabriel Stokes||, in 1845, to involve no other assumption than that the stresses, other than that of pressure uniform in all directions, are linear functions of the rates of distortion, with a coefficient depending on the physical state of the fluid.

By obtaining a singular solution of these equations as applied to the case of pendulums in steady periodic motion, Sir G. Stokes¶ was able to compare the theoretical results with the numerous experiments that had

* *Mém. de l'Académie*, vol. VI. p. 389.
† *Mém. des Savants Étrangers*, vol. I. p. 40.
‡ *Mém. de l'Académie*, vol. X. p. 345.
§ *B.A. Report*, 1846.
|| *Cambridge Phil. Trans.*, 1845.
¶ *Ibid.*, vol. IX. 1857.

This reprint of Reynolds's original paper is from his *Papers on mechanical and physical subjects*, vol. II, pp. 535–577 (1901, Cambridge University Press), and is an emended version of the first printing, which appeared in *Phil. Trans. R. Soc. Lond.* A **186**, 123–164 (1895).

been recorded, with the result that the theoretical calculations agreed so closely with the experimental determinations as seemingly to prove the truth of the assumption involved. This was also the result of comparing the flow of water through uniform tubes with the flow calculated from a singular solution of the equations, so long as the tubes were small and the velocities slow. On the other hand, these results, both theoretical and practical, were directly at variance with common experience as to the resistance encountered by larger bodies moving with higher velocities through water, or by water moving with greater velocities through larger tubes. This discrepancy Sir G. Stokes considered as probably resulting from eddies, which rendered the actual motion other than that to which the singular solution referred, and not as disproving the assumption.

In 1850, after Joule's discovery of the Mechanical Equivalent of Heat, Stokes showed, by transforming the equations of motion—with arbitrary stresses—so as to obtain the equations of ("Vis-viva") energy, that this equation contained a definite function, which represented the difference between the work done on the fluid by the stresses and the rate of increase of the energy, per unit of volume, which function, he concluded, must, according to Joule, represent the Vis-viva converted into heat.

This conclusion was obtained from the equations irrespective of any particular relation between the stresses and the rates of distortion. Sir G. Stokes, however, translated the function into an expression in terms of the rates of distortion, which expression has since been named by Lord Rayleigh the *Dissipation-Function*.

2. In 1883 I succeeded in proving, by means of experiments with colour bands—the results of which were communicated to the Society*—that when water is caused by pressure to flow through a uniform smooth pipe, the motion of the water is *direct*, *i.e.*, parallel to the sides of the pipe, or *sinuous*, *i.e.*, crossing and re-crossing the pipe, according as U_m, the mean velocity of the water, as measured by dividing Q, the discharge, by Δ, the area of the section of the pipe, is below or above a certain value given by

$$K\mu/D\rho,$$

where D is the diameter of the pipe, ρ the density of the water, and K a numerical constant, the value of which according to my experiments, and, as I was able to show, to all the experiments by Poiseuille and Darcy, is for pipes of circular section between

$$1900 \text{ and } 2000,$$

* *Phil. Trans.*, 1883, Part III. p. 935. (See this vol. p. 51.)

or, in other words, steady direct motion in round tubes is stable or unstable according as

$$\rho \frac{DU_m}{\mu} > 1900 \text{ or } < 2000,$$

the number K being thus a criterion of the possible maintenance of sinuous or eddying motion.

3. The experiments also showed that K was equally a criterion of the law of the resistance to be overcome—which changes from a resistance proportional to the velocity, and in exact accordance with the theoretical results obtained from the singular solution of the equation, when direct motion changes to sinuous, *i.e.*, when

$$\rho \frac{DU_m}{\mu} = K.$$

4. In the same paper I pointed out that the existence of this sudden change in the law of motion of fluids between solid surfaces when

$$DU_m = \frac{\mu}{\rho} K$$

proved the dependence of the manner of motion of the fluid on a relation between the product of the dimensions of the pipe multiplied by the velocity of the fluid, and the product of the molecular dimensions multiplied by the molecular velocities which determine the value of

$$\mu$$

for the fluid, also that the equations of motion for viscous fluid contained evidence of this relation.

These experimental results completely removed the discrepancy previously noticed, showing that, whatever may be the cause, in those cases in which the experimental results do not accord with those obtained by the singular solution of the equations, the actual motions of the water are different. But in this there is only a partial explanation, for there remains the mechanical or physical significance of the existence of the criterion to be explained.

5. [My object in this paper is to show that the theoretical existence of an inferior limit to the criterion follows from the equations of motion as a consequence:—

(1) Of a more rigorous examination and definition of the geometrical basis on which the analytical method of distinguishing between molar-motions and heat-motions in the kinetic theory of matter is founded; and

(2) Of the application of the same method of analysis, thus definitely founded, to distinguish between mean-molar-motions and relative-molar-motions, where, as in the case of steady-mean-flow along a pipe, the more rigorous definition of the geometrical basis shows the method to be strictly applicable, and in other cases where it is approximately applicable.

The geometrical relation of the motions respectively indicated by the terms mean-molar-, or MEAN-MEAN-MOTION, and relative-molar-, or RELATIVE-MEAN-MOTION, being essentially the same as the relation of the respective motions indicated by the terms molar-, or MEAN-MOTION, and relative-, or HEAT-MOTION, as used in the theory of gases.

I also show that the limit to the criterion obtained by this method of analysis, and by integrating the equations of motion in space, appears as a *geometrical limit* to the *possible simultaneous distribution of certain quantities in space,* and in no wise depends on the physical significance of these quantities. Yet the physical significance of these quantities, as defined in the equations, becomes so clearly exposed as to indicate that further study of the equations would elucidate the properties of matter and mechanical principles involved, and so be the means of explaining what has hitherto been obscure in the connection between thermodynamics and the principles of mechanics.

The geometrical basis of the method of analysis used in the kinetic theory of gases has hitherto consisted:—

(1) Of the geometrical *principle* that the motion of any point of a mechanical system may, at any instant, be abstracted into the mean-motion of the whole system at that instant, and the motion of the point relative to the mean-motion; and

(2) Of the *assumption* that the component, in any particular direction, of the velocity of a molecule, may be abstracted into a mean-component-velocity (say u) which is the mean-component-velocity of all the molecules in the immediate neighbourhood, and a relative-velocity (say ξ), which is the difference between u and the component-velocity of the molecule*; u and ξ being so related that, M being the mass of the molecule, the integrals of $(M\xi)$, and $(Mu\xi)$, &c., over all the molecules in the immediate neighbourhood are zero, and $\Sigma\,[M\,(u+\xi)^2] = \Sigma\,[M\,(u^2+\xi^2)]$†.

The geometrical principle (1) has only been used to distinguish between the energy of the mean-motion of the molecule, and the energy of its internal motions taken relatively to its mean-motion; and so to eliminate the internal motions from all further geometrical considerations which rest on the assumption (2).

* "Dynamical Theory of Gases," *Phil. Trans.*, 1866, p. 67.

† *Phil. Trans.*, 1866, p. 71.

That this assumption (2) is purely geometrical, becomes at once obvious, when it is noticed that the argument relates solely to the distribution in space of certain quantities at a particular instant of time. And it appears that the questions as to whether the assumed distinctions are possible under any distributions, and, if so, under what distribution, are proper subjects for geometrical solution.

On putting aside the apparent obviousness of the assumption (2), and considering definitely what it implies, the necessity for further definition at once appears.

The mean-component-velocity (u) of all the molecules in the immediate neighbourhood of a point, say P, can only be the mean-component-velocity of all the molecules in some space (S) enclosing P. u is then the mean-component-velocity of the mechanical system enclosed in S, and, for this system, is the mean-velocity at every point within S, and, multiplied by the entire mass within S, is the whole component momentum of the system. But according to the assumption (2), u with its derivatives are to be continuous functions of the position of P, which functions may vary from point to point even within S; so that u is not taken to represent the mean-component-velocity of the system within S, but the mean-velocity at the point P. Although there seems to have been no specific statement to that effect, it is presumable that the space S has been assumed to be so taken that P is the centre of gravity of the system within S. The relative positions of P and S being so defined, the shape and size of the space S requires to be further defined, so that u, &c., may vary continuously with the position of P, which is a condition that can always be satisfied if the size and shape of S may vary continuously with the position of P.

Having thus defined the relation of P to S and the shape and size of the latter, expressions may be obtained for the conditions of distribution of u, for which $\Sigma(M\xi)$ taken over S will be zero, *i.e.*, for which the condition of mean-momentum shall be satisfied.

Taking S_1, u_1, &c., as relating to a point P_1 and S, u, &c., as relating to P, another point, of which the component distances from P_1 are x, y, z; P_1 is the C.G. of S_1, and by however much or little S may overlap S_1, S has its centre of gravity at x, y, z, and is so chosen that u, &c., may be continuous functions of x, y, z; u may, therefore, differ from u_1 even if P is within S_1. Let u be taken for every molecule of the system S_1. Then according to assumption (2), $\Sigma(Mu)$ over S_1 must represent the component of momentum of the system within S_1, that is, in order to satisfy the condition of mean-momentum, the mean-value of the variable quantity u over the system S_1 must be equal to u_1 the mean-component-velocity of the system S_1, and this is a condition which, in consequence of the geometrical definition already mentioned, can only be satisfied under certain distributions of u. For since u is a continuous function of x, y, z, $M(u-u_1)$ may be expressed as a function of the derivatives of u at P_1 multiplied by corresponding powers and products of x, y, z, and again by M; and by equating the integral of this function over the space S_1 to zero, a definite expression is obtained, in terms of the limits imposed on x, y, z, by the already-defined space S_1 for the geometrical condition as to the distribution of u under which the condition of mean-momentum can be satisfied.

From this definite expression it appears, as has been obvious all through the argument, that the condition is satisfied if u is constant. It also appears that there are certain other well-defined systems of distribution for which the condition is strictly satisfied, and that for all other distributions of u the condition of mean-momentum can only be approximately satisfied to a degree for which definite expressions appear.

Having obtained the expression for the condition of distribution of u, so as to satisfy the condition of mean-momentum, by means of the expression for $M(u-u')$, &c., expressions are obtained for the conditions as to the distribution of ξ, &c., in order that the integrals over the space S_1 of the products $M(u\xi)$, &c. may be zero when $\Sigma\,[M\,(u-u_1)]=0$, and the conditions of mean-energy satisfied as well as those of mean-momentum. It then appears that in some particular cases of distribution of u, under which the condition of mean-momentum is strictly satisfied, certain conditions as to the distribution of ξ, &c., must be satisfied in order that the energies of mean- and relative-motion may be distinct. These conditions as to the distribution of ξ, &c., are, however, obviously satisfied in the case of heat-motion, and do not present themselves otherwise in this paper.

From the definite geometrical basis thus obtained, and the definite expressions which follow for the condition of distribution of u, &c., under which the method of analysis is strictly applicable, it appears that this method may be rendered generally applicable to any system of motion by a slight adaptation of the meaning of the symbols, and that it does not necessitate the elimination of the internal motion of the molecules, as has been the custom in the theory of gases.

Taking u, v, w to represent the motions (continuous or discontinuous) of the matter passing a point, and ρ to represent the density at the point, and putting $\bar{u}$, &c., for the mean-motion (instead of u as above), and u', &c., for the relative-motion (instead of ξ as before), the geometrical conditions as to the distribution of $\bar{u}$, &c., to satisfy the conditions of mean-momentum and mean-energy are, substituting ρ for M, of precisely the same form as before, and as thus expressed, the theorem is applicable to any mechanical system however abstract.

(1) In order to obtain the conditions of distribution of molar-motion, under which the condition of mean-momentum will be satisfied, so that the energy of molar-motion may be separated from that of the heat-motion, u, &c., and ρ are taken as referring to the actual motion and density at a point in a molecule, and S_1 is taken of such dimensions as may correspond to the scale, or periods in space, of the molecular distances, then the conditions of distribution of $\bar{u}$, under which the condition of mean-momentum is satisfied, become the conditions as to the distribution of molar-motion, under which it is possible to distinguish between the energies of molar-motions and heat-motions.

(2) And, when the conditions in (1) are satisfied to a sufficient degree of approximation by taking u to represent the molar-motion ($\bar{u}$ in (1)), and the dimensions of the space S to correspond with the period in space or scale of any possible periodic or eddying motion, the conditions as to the distribution of $\bar{u}$, &c. (the components of mean-mean-motion), which satisfy the condition of mean-momentum, show the conditions of mean-molar-motion, under which it is possible to separate the energy of mean-molar-motion from the energy of relative-molar- (or relative-mean-) motion.

Having thus placed the analytical method used in the kinetic theory on a definite geometrical basis, and adapted so as to render it applicable to all systems of motion, by applying it to the dynamical theory of viscous fluid, I have been able to show:—Feb. 18, 1895.]

(*a*) That the adoption of the conclusion arrived at by Sir Gabriel Stokes, that the dissipation function represents the rate at which heat is produced, adds a definition to the meaning of u, v, w—the components of mean or fluid velocity—which was previously wanting.

(*b*) That as the result of this definition the equations are true, and are only true, as applied to fluid in which the mean-motions of the matter, excluding the heat-motions, are steady.

(*c*) That the evidence of the possible existence of such steady mean-motions, while at the same time the conversion of the energy of these mean-motions into heat is going on, proves the existence of some *discriminative cause*, by which the *periods* in space and time of the mean-motion are prevented from approximating in magnitude to the corresponding *periods* of the heat-motions, and also proves the existence of some general action by which the energy of mean-motion is continually *transformed* into the energy of heat-motion, without passing through any intermediate stage.

(*d*) That as applied to fluid in unsteady mean-motion (excluding the heat-motions), however steady the mean integral flow may be, the equations are approximately true in a degree which increases with the ratios of the magnitudes of the *periods*, in time and space, of the mean-motion, to the magnitude of the corresponding periods of the heat-motions.

(*e*) That if the *discriminative cause* and the *action of transformation* are the result of general properties of matter, and not of properties which affect only the ultimate motions, there must exist evidence of similar actions as between the mean-mean-motion, in directions of mean-flow, and the periodic mean-motions taken relative to the mean-mean-motion but excluding heat-motions. And that such evidence must be of a general and important kind, such as the unexplained laws of the resistance of fluid motions, the law of the universal dissipation of energy, and the second law of thermodynamics.

(*f*) That the *generality* of the effects of the properties on which the *action of transformation* depends, is proved by the fact that resistance, other than proportional to the velocity, is caused by the relative (eddying) mean-motion.

(*g*) That the existence of the *discriminative cause* is directly proved by the existence of the *criterion*, the dependence of which on circumstances which limit the magnitudes of the periods of relative-mean-motion, as compared with the heat-motion, also proves the *generality* of the effects of the properties on which it depends.

(*h*) That the proof of the generality of the effects of the properties on which the discriminative cause, and the action of transformation depend, shows that—if in the equations of motion the mean-mean-motion is distinguished from the relative-mean-motion in the same way as the mean-motion is distinguished from the heat-motions—(1) the equations must contain expressions for the *transformation* of the energy of mean-mean-motion to energy of relative-mean-motion; and (2) that the equations, when integrated over a complete system, must show that the possibility of relative-mean-motion depends on the ratio of the possible magnitudes of the periods of relative-mean-motion, as compared with the corresponding magnitude of the periods of the heat-motions.

(*i*) That when the equations are transformed so as to distinguish between the mean-mean-motions, of infinite periods, and the relative-mean-motions of finite periods, there result two distinct systems of equations, one system for mean-mean-motion, as affected by relative-mean-motion and heat-motion, the other system for relative-mean-motion as affected by mean-mean-motion and heat-motions.

(*j*) That the equation of energy of mean-mean-motion, as obtained from the first system, shows that the rate of increase of energy is diminished by conversion into heat, and by transformation of energy of mean-mean-motion in consequence of the relative-mean-motion, which transformation is expressed by a function identical in form with that which expresses the conversion into heat; and that the equation of energy of relative-mean-motion, obtained from the second system, shows that this energy is increased only by transformation of energy from mean-mean-motion expressed by the same function, and diminished only by the conversion of energy of relative-mean-motion into heat.

(*k*) That the difference of the two rates (1) transformation of energy of mean-mean-motion into energy of relative-mean-motion as expressed by the transformation function, (2) the conversion of energy of relative-mean-motion into heat, as expressed by the function expressing dissipation of the energy of relative-mean-motion, affords a discriminating equation as to the conditions under which relative-mean-motion can be maintained.

(*l*) That this discriminating equation is independent of the energy of relative-mean-motion, and expresses a relation between variations of mean-mean-motion of the first order, the space periods of relative-mean-motion, and μ/ρ, such that any circumstances which determine the maximum periods of the relative-mean-motion, determine the conditions of mean-mean-motion under which relative-mean-motion will be maintained, that is, determine *the criterion.*

(*m*) That as applied to water in steady mean-flow between parallel plane surfaces, the boundary conditions, and the equation of continuity, impose limits to the maximum space periods of relative-mean-motion, such that the discriminating equation affords definite proof that when an indefinitely small sinuous or relative disturbance exists, it must fade away if

$$\rho D U_m/\mu$$

is less than a certain number, which depends on the shape of the section of the boundaries, and is constant as long as there is geometrical similarity. While for greater values of this function, in so far as the discriminating equation shows, the energy of sinuous motion may increase until it reaches to a definite limit, and rules the resistance.

(*n*) That besides thus affording a mechanical explanation of the existence of the criterion K, the discriminating equation shows the purely geometrical circumstances on which the value of K depends, and although these circumstances must satisfy geometrical conditions required for steady mean-motion other than those imposed by the conservations of mean-energy and momentum, the theory admits of the determination of an inferior limit to the value of K under any definite boundary conditions, which, as determined for the particular case, is

$$517.$$

This is below the experimental value for round pipes, and is about half what might be expected to be the experimental value for a flat pipe, which leaves a margin to meet the other kinematical conditions for steady mean-mean-motion.

(*o*) That the discriminating equation also affords a definite expression for the resistance, which proves that, with smooth fixed boundaries, the conditions of dynamical similarity under any geometrical similar circumstances depend only on the value of

$$\frac{\rho}{\mu^2}\frac{dp}{dx}b^3,$$

where b is one of the lateral dimensions of the pipe; and that the expression for this resistance is complex, but shows that above the critical velocity the relative-mean-motion is limited, and that the resistances increase as a power of the velocity higher than the first.

Section II.

The Mean-motion and Heat-motions as distinguished by Periods.—Mean-mean-motion and Relative-mean-motion.—Discriminative Cause and Action of Transformation.—Two Systems of Equations.—A Discriminating Equation.

6. Taking the general equations of motion for incompressible fluid, subject to no external forces to be expressed by

$$\left.\begin{aligned}\rho\frac{du}{dt} &= -\left\{\frac{d}{dx}(p_{xx}+\rho uu)+\frac{d}{dy}(p_{yx}+\rho uv)+\frac{d}{dz}(p_{zx}+\rho uw)\right\}\\ \rho\frac{dv}{dt} &= -\left\{\frac{d}{dx}(p_{xy}+\rho vu)+\frac{d}{dy}(p_{yy}+\rho vv)+\frac{d}{dz}(p_{zy}+\rho vw)\right\}\\ \rho\frac{dw}{dt} &= -\left\{\frac{d}{dx}(p_{xz}+\rho wu)+\frac{d}{dy}(p_{yz}+\rho wv)+\frac{d}{dz}(p_{zz}+\rho ww)\right\}\end{aligned}\right\}\ldots\ldots(1),$$

with the equation of continuity

$$0 = du/dx + dv/dy + dw/dz\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where p_{xx}, &c., are arbitrary expressions for the component forces per unit of area, resulting from the stresses, acting on the negative faces of planes perpendicular to the direction indicated by the first suffix, in the direction indicated by the second suffix.

Then multiplying these equations respectively by u, v, w, integrating by parts, adding and putting

$$2E \text{ for } \rho\,(u^2+v^2+w^2)$$

and transposing, the rate of increase of kinetic energy per unit of volume is given by

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)E=-\left\{\begin{aligned}&\frac{d}{dx}(up_{xx})+\frac{d}{dy}(up_{yx})+\frac{d}{dz}(up_{zx})\\&+\frac{d}{dx}(vp_{xy})+\frac{d}{dy}(vp_{yy})+\frac{d}{dz}(vp_{zy})\\&+\frac{d}{dx}(wp_{xz})+\frac{d}{dy}(wp_{yz})+\frac{d}{dz}(wp_{zz})\end{aligned}\right\}$$

$$+\left\{\begin{aligned}&p_{xx}\frac{du}{dx}+p_{yx}\frac{du}{dy}+p_{zx}\frac{du}{dz}\\&+p_{xy}\frac{dv}{dx}+p_{yy}\frac{dv}{dy}+p_{zy}\frac{dv}{dz}\\&+p_{xz}\frac{dw}{dx}+p_{yz}\frac{dw}{dy}+p_{zz}\frac{dw}{dz}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(3).$$

The left member of this equation expresses the *rate of increase* in the kinetic energy of the fluid per unit of volume at a point moving with the fluid.

The first term on the right expresses the *rate at which work* is being done by the surrounding fluid per unit of volume at a point.

The second term on the right therefore, by the law of conservation of energy, expresses the difference between the rate of increase of kinetic energy and the rate at which work is being done by the stresses. This difference has, so far as I am aware, in the absence of other forces, or any changes of potential energy, been equated to the rate at which heat is being converted into energy of motion, Sir Gabriel Stokes having first indicated this* as resulting from the law of conservation of energy then just established by Joule.

7. This conclusion, that the second term on the right of (3) expresses the rate at which heat is being converted, as it is usually accepted, may be correct enough, but there is a consequence of adopting this conclusion which enters largely into the method of reasoning in this paper, but which, so far as I know, has not previously received any definite notice.

* *Cambridge Phil. Trans.*, vol. IX. p. 57.

The Component Velocities in the Equations of Viscous Fluids.

In no case, that I am aware of, has any very strict definition of u, v, w, as they occur in the equations of motion, been attempted. They are usually defined as the velocities of a particle at a point (x, y, z) of the fluid, which may mean that they are the actual component-velocities of the point in the matter passing at the instant, or that they are the mean-velocities of all the matter in some space enclosing the point, or which passes the point in an interval of time. If the first view is taken, then the right-hand member of the equation represents the rate of increase of kinetic energy, per unit of volume, in the matter at the point; and the integral of this expression over any finite space S, moving with the fluid, represents the total rate of increase of kinetic energy, including heat-motion, within that space; hence the difference between the rate at which work is done on the surface of S, and the rate at which kinetic energy is increasing can, by the law of conservation of energy, only represent the rate at which that part of the heat which does not consist in kinetic energy of matter is being produced, whence it follows:—

(*a*) *That the adoption of the conclusion that the second term in equation* (3) *expresses the rate at which heat is being converted, defines u, v, w, as not representing the component-velocities of points in the passing matter.*

Further, if it is understood that u, v, w, represent the mean velocities of the matter in some space, enclosing x, y, z, the point considered, or the mean-velocities at a point taken over a certain interval of time, so that $\Sigma(\rho u)$, $\Sigma(\rho v)$, $\Sigma(\rho w)$ may express the components of momentum, and $z\Sigma(\rho v) - y\Sigma(\rho w)$, &c., &c., may express the components of moments of momentum, of the matter over which the mean is taken; there still remains the question as to what spaces and what intervals of time.

(*b*) *Hence the conclusion that the second term expresses the rate of conversion of heat, defines the spaces and intervals of time over which the mean-component-velocities must be taken, so that E may include all the energy of mean-motion, and exclude that of heat-motions.*

Equations Approximate only except in Three Particular Cases.

8. According to the reasoning of the last article, if the second term on the right of equation (3) expresses the rate at which heat is being converted into energy of mean-motion, either ρu, ρv, ρw express the mean components of momentum of the matter, taken at any instant over a space S_0 enclosing the point x, y, z, to which u, v, w refer, so that this point is the centre of gravity of the matter within S_0 and such that ρ represents the mean density of the matter within this space; or ρu, ρv, ρw represent the mean components of momentum taken at x, y, z over an interval of time τ, such that ρ is

the mean density over the time τ, and if t marks the instant to which u, v, w refer, and t' any other instant, $\Sigma[(t-t')\rho]$, in which ρ is the actual density, taken over the interval τ is zero. The equations, however, require, that so obtained, ρ, u, v, w shall be continuous functions of space and time, and it can be shown that this involves certain conditions between the distribution of the mean-motion and the dimensions of S_0 and τ.

Mean- and Relative-motions of Matter.

Whatever the motions of matter within a fixed space S may be at any instant, if the component-velocities at a point are expressed by u, v, w, the mean-component-velocities taken over S will be expressed by

$$\bar{u} = \frac{\Sigma(\rho u)}{\Sigma(\rho)}, \text{ \&c., \&c. } \qquad\qquad (4).$$

If then $\bar{u}, \bar{v}, \bar{w}$ are taken at each instant as the velocities of x, y, z, the *instantaneous centre of gravity of the matter within* S, the component momentum at the centre of gravity may be put

$$\rho u = \rho\bar{u} + \rho u' \qquad\qquad (5),$$

where u' is the motion of the matter, relative to axes moving with the mean velocity, at the centre of gravity of the matter within S. Since a space S of definite size and shape may be taken about any point x, y, z in an indefinitely larger space, so that x, y, z is the centre of gravity of the matter within S, the motion in the larger space may be divided into two distinct systems of motion, of which $\bar{u}, \bar{v}, \bar{w}$ represent a mean-motion at each point and u', v', w' a motion at the same point relative to the mean-motion at the point.

If, however, $\bar{u}, \bar{v}, \bar{w}$ are to represent the real mean-motion, it is necessary that $\Sigma(\rho u')$, $\Sigma(\rho v')$, $\Sigma(\rho w')$ summed over the space S, taken about any point, shall be severally zero; and in order that this may be so, certain conditions must be fulfilled.

For taking x, y, z, for G, the centre of gravity of the matter within S, and x', y', z' for any other point within S, and putting a, b, c for the dimensions of S in directions x, y, z, measured from the point x, y, z; since $\bar{u}, \bar{v}, \bar{w}$ are continuous functions of x, y, z by shifting S so that the centre of gravity of the matter within it is at x', y', z', the value of $\bar{u}$ for this point is given by

$$\bar{u} = \bar{u}_G + (x'-x)\left(\frac{d\bar{u}}{dx}\right)_G + (y'-y)\left(\frac{d\bar{u}}{dy}\right)_G + (z'-z)\left(\frac{d\bar{u}}{dz}\right)_G + \tfrac{1}{2}(x'-x)^2\left(\frac{d^2\bar{u}}{dx^2}\right)_G + \text{\&c.} \qquad (6),$$

where all the differential coefficients on the left refer to the point x, y, z; and in the same way for $\bar{v}$ and $\bar{w}$.

Subtracting the value of $\bar{u}$ thus obtained for the point x', y', z', from that

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of u at the same point, the difference is the value of u' at this point, whence summing these differences over the space S about G at x, y, z, since by definition when summed over the space S about G

$$\Sigma\left[\rho\left(u-\bar{u}_G\right)\right]=0 \text{ and } \Sigma\left[\rho\left(x'-x\right)\right]=0 \quad\ldots\ldots\ldots\ldots\ldots (7),$$

$$\left.\begin{array}{l}\Sigma(\rho u')=-\left\{\frac{1}{2}\Sigma\left[\rho(x-x')^2\right]\left(\dfrac{d^2\bar{u}}{dx^2}\right)_G+\frac{1}{2}\Sigma\left[\rho(y-y')^2\right]\left(\dfrac{d^2\bar{u}}{dy^2}\right)_G\right. \\ \qquad\qquad\qquad\qquad\qquad \left.+\frac{1}{2}\Sigma\left[\rho(z-z')^2\right]\left(\dfrac{d^2\bar{u}}{dz^2}\right)_G+\&c.\right\} \\ \text{That is} \\ \dfrac{\Sigma(\rho u')}{\Sigma(\rho)} \text{ is } < -\left\{\dfrac{a^2}{2}\left(\dfrac{d^2\bar{u}}{dx^2}\right)_G+\dfrac{b^2}{2}\left(\dfrac{d^2\bar{u}}{dy^2}\right)_G+\dfrac{c^2}{2}\left(\dfrac{d^2\bar{u}}{dz^2}\right)_G+\&c.\right\}\end{array}\right\} \ldots\ldots (8\text{A}).$$

In the same way if Σ be taken over the interval of time τ including t; and for the instant t

$$\bar{u}=\frac{\Sigma(\rho u)}{\Sigma(\rho)}, \text{ and } \rho\bar{u}=\rho u+\rho u';$$

then since for any other instant t'

$$\bar{u}=\bar{u}_t+(t-t')\left(\frac{d\bar{u}}{dt}\right)_t+\tfrac{1}{2}(t-t')^2\left(\frac{d^2\bar{u}}{dt^2}\right)_t+\&c.,$$

where $\Sigma\left[\rho\left(t-t'\right)\right]=0$, and $\Sigma\left[\rho\left(\bar{u}_t-u\right)\right]=0$.

It appears that

$$\left.\begin{array}{l}\Sigma(\rho u')=-\Sigma\left[\frac{1}{2}\rho(t-t')^2\right]\dfrac{d^2u}{dt^2}+\&c. \\ \dfrac{\Sigma(\rho u')}{\Sigma(\rho)} \text{ is } < -\frac{1}{2}\tau^2\left(\dfrac{d^2\bar{u}}{dt^2}\right)_t-\&c.\end{array}\right\} \ldots\ldots\ldots\ldots\ldots (8\text{B}).$$

From equations (8A) and (8B), and similar equations for $\Sigma(\rho v')$ and $\Sigma(\rho w')$, it appears that if

$$\Sigma(\rho u')=\Sigma(\rho v')=\Sigma(\rho w')=0,$$

where the summation extends both over the space S and the interval τ, all the terms on the right of equations (8A) and (8B) must be respectively and continuously zero, or, what is the same thing, all the differential coefficients of $\bar{u}$, $\bar{v}$, $\bar{w}$ with respect to x, y, z and t of the first order must be respectively constant.

This condition will be satisfied if the mean-motion is steady, or uniformly varying with the time, and is everywhere in the same direction, being subject to no variations in the direction of motion; for suppose the direction of motion to be that of x, then since the periodic motion passes through a complete period within the distance $2a$, $\Sigma(\rho u')$ will be zero within the space

$$2a\,.\,dy\,.\,dz,$$

however small $dy \,.\, dz$ may be, and since the only variations of the mean-motion are in directions y and z, in which b and c may be taken zero, and du/dt is everywhere constant, the conditions are perfectly satisfied.

The conditions are also satisfied if the mean-motion is that of uniform expansion or contraction, or is that of a rigid body.

These three cases, in which it may be noticed that variations of mean-motion are everywhere uniform in the direction of motion, and subject to steady variations in respect of time, are the only cases in which the conditions (8 A), (8 B), can be perfectly satisfied.

The conditions will, however, be approximately satisfied, when the variations of $\bar{u}$, $\bar{v}$, $\bar{w}$ of the first order are approximately constant over the space S.

In such case the right-hand members of equations (8 A), (8 B), are neglected, and it appears that the closeness of the approximations will be measured by the relative magnitude of such terms as

$$a\,\frac{d^2\bar{u}}{dx^2},\ \&c.,\ \tau\,\frac{d^2\bar{u}}{dt^2}\ \text{as compared with}\ \frac{d\bar{u}}{dx},\ \frac{d\bar{u}}{dt},\ \&c.$$

Since frequent reference must be made to these relative values, and, as in periodic motion, the relative values of such terms are measured by the period (in space or time) as compared with a, b, c and τ, which are, in a sense, the periods of u', v', w', I shall use the term period in this sense, taking note of the fact that when the mean-motion is constant in the direction of motion, or varies uniformly in respect of time, it is not periodic, *i.e.*, its periods are infinite.

9. It is thus seen that the closeness of the approximation with which the motion of any system can be expressed as a varying mean-motion together with a relative-motion, which, when integrated over a space of which the dimensions are a, b, c, has no momentum, increases as the magnitude of the periods of $\bar{u}$, $\bar{v}$, $\bar{w}$ in comparison with the periods of u', v', w', and is measured by the ratio of the relative orders of magnitudes to which these periods belong.

Heat-motions in Matter are Approximately Relative to the Mean-motions.

The general experience that heat in no way affects the momentum of matter, shows that the heat-motions are relative to the mean-motions of matter taken over spaces of sensible size. But, as heat is by no means the only state of relative-motion of matter, if the heat-motions are relative to all mean-motions of matter, whatsoever their periods may be, it follows—that there must be some *discriminative cause* which prevents the existence of relative-motions of matter other than heat, except mean-motions with periods in time and space of greatly higher orders of magnitude than the corresponding periods of the heat-motions—otherwise, by equations (8A), (8B), heat-motions could not be to a high degree of approximation relative to all other motions, and we could not have to a high degree of approximation,

$$\left.\begin{array}{l} p_{xx}\dfrac{du}{dx}+p_{yx}\dfrac{du}{dy}+p_{zx}\dfrac{du}{dz} \\ p_{xy}\dfrac{dv}{dx}+p_{yy}\dfrac{dv}{dy}+p_{zy}\dfrac{dv}{dz} \\ p_{xz}\dfrac{dw}{dx}+p_{yz}\dfrac{dw}{dy}+p_{zz}\dfrac{dw}{dz} \end{array}\right\} = -\frac{d}{dt}(pH) \quad \ldots\ldots\ldots\ldots\ldots (9),$$

where the expression on the right stands for the rate at which heat is converted into energy of mean-motion.

Transformation of Energy of Relative-mean-motion to Energy of Heat-motion.

10. The recognition of the existence of a *discriminative cause*, which prevents the existence of relative-mean-motions with periods of the same order of magnitude as heat-motions, proves the existence of another general action by which the energy of relative-mean-motion, of which the periods are of another and higher order of magnitude than those of the heat-motions, is *transformed* to energy of heat-motion.

For if relative-mean-motions cannot exist with periods approximating to those of heat, the conversion of energy of mean-motion into energy of heat, proved by Joule, cannot proceed by the gradual degradation of the periods of mean-motion until these periods coincide with those of heat, but must, in its final stages, at all events, be the result of some action which causes the energy of relative-mean-motion to be transformed into the energy of heat-motions, without intermediate existence in states of relative-motion, with intermediate and gradually diminishing periods.

That such change of energy of mean-motion to energy of heat may be properly called transformation, becomes apparent when it is remembered that neither mean-motion nor relative-motion has any separate existence, but are only abstract quantities, determined by the particular process of abstraction, and so changes in the actual-motion may, by the process of abstraction, cause transformation of the abstract energy of the one abstract-motion, to abstract energy of the other abstract-motion.

All such transformation must depend on the changes in the actual-motions, and so must depend on mechanical principles and the properties of matter, and hence the direct passage of energy of relative-mean-motion to energy of heat-motions is evidence of a general cause of the condition of actual-motion which results in transformation—which may be called *the cause of transformation.*

The Discriminative Cause, and the Cause of Transformation.

11. The only known characteristic of heat-motions, besides that of being relative to the mean-motion, already mentioned, is that the motions of matter which result from heat are an ultimate form of motion which does not alter so long as the mean-motion is uniform over the space, and so long as no change of state occurs in the matter. In respect of this characteristic, heat-motions are, so far as we know, unique, and it would appear that heat-motions are distinguished from the mean-motions by some ultimate properties of matter.

It does not, however, follow that the cause of transformation, or even the discriminative cause, are determined by these properties. Whether this is so or not can only be ascertained by experience. If either or both these causes depend solely on properties of matter which only affect the heat-motions, then no similar effect would result as between the variations of mean-mean-motion and relative-mean-motion, whatever might be the difference in magnitude of their respective periods. Whereas, if these causes depend on properties of matter which affect all modes of motion, distinctions in periods must exist between mean-mean-motion and relative-mean-motion, and transformation of energy take place from one to the other, as between the mean-motion and the heat-motions.

The mean-mean-motion cannot, however, under any circumstances stand to the relative-mean-motion in the same relation as the mean-motion stands to the heat-motions, because the heat-motions cannot be absent, and in addition to any transformation from mean-mean-motion to relative-mean-motion, there are transformations both from mean- and relative-mean-motion to heat-motions, which transformation may have important effects on both the transformation of energy from mean- to relative-mean-motion, and on the discriminative cause of distinction in their periods.

In spite of the confusing effect of the ever present heat-motions, it would, however, seem that evidence as to the character of the properties on which the cause of transformation and the discriminative cause depend, should be forthcoming as the result of observing the mean- and relative-mean-motions of matter.

12. To prove by experimental evidence that the effects of these properties of matter are confined to the heat-motions, would be to prove a negative; but if these properties are in any degree common to all modes of matter, then at first sight it must seem in the highest degree improbable that the effects of these causes on the mean- and relative-mean-motions

would be obscure, and only to be observed by delicate tests. For properties which can cause distinctions between the mean- and heat-motions of matter so fundamental and general, that from the time these motions were first recognized the distinction has been accepted as part of the order of nature, and has been so familiar to us that its cause has excited no curiosity, must, if they have any effect at all, cause effects which are general and important on the mean-motions of matter. It would thus seem that evidence of the general effects of such properties should be sought in those laws and phenomena known to us as the result of experience, but of which no rational explanation has hitherto been found; such as the law that the resistance of fluids moving between solid surfaces and of solids moving through fluids, in such a manner that the general-motion is not periodic, is as the square of the velocities, the evidence covered by the law of the universal tendency of all energy to dissipation, and the second law of thermodynamics.

13. In considering the first of the instances mentioned, it will be seen that the evidence it affords as to the general effect of the properties, on which depends transformation of energy from mean- to relative-motion, is very direct. For, since my experiments with colour bands have shown that when the resistance of fluids, in steady mean flow, varies with a power of the velocity higher than the first, the fluid is always in a state of sinuous motion, it appears that the prevalence of such resistance is evidence of the existence of a general action, by which energy of mean-mean-motion, with infinite periods, is directly transformed to the energy of relative-mean-motion, with finite periods, represented by the eddying motion, which renders the general mean-motion sinuous, by which transformation the state of eddying-motion is maintained, notwithstanding the continual transformation of its energy into heat-motions.

We have thus direct evidence that properties of matter which determine the cause of transformation, produce general and important effects which are not confined to the heat-motions.

In the same way, the experimental demonstration I was able to obtain, that relative-mean-motion in the form of eddies of finite periods, both as shown by colour bands and as shown by the law of resistances, cannot be maintained except under circumstances depending on the conditions which determine the superior limits to the velocity of the mean-mean-motion, of infinite periods, and the periods of the relative-mean-motion, as defined in the criterion

$$DU_m/\mu = K \qquad (10),$$

is not only a direct experimental proof of the existence of a discriminative cause which prevents the maintenance of periodic mean-motion except with periods greatly in excess of the periods of the heat-motions, but also indicates that the discriminative cause depends on properties of matter which affect the mean-motions as well as the heat-motions.

Expressions for the Rate of Transformation and the Discriminative Cause.

14. It has already been shown (Art. 8) that the equations of motion approximate to a true expression of the relations between the mean-motions and stresses, when the ratio of the periods of mean-motions to the periods of the heat-motions approximates to infinity. Hence it follows that these equations must of necessity include whatever mechanical or kinematical principles are involved in the transformation of energy of mean-mean-motion to energy of relative-mean-motion. It has also been shown that the properties of matter, on which depends the transformation of energy of varying mean-motion to relative-motion, are common to the relative-mean-motion as well as to the heat-motion. Hence, if the equations of motion are applied to a condition in which the mean-motion consists of two components, the one component being a mean-mean-motion, as obtained by integrating the mean-motion over spaces S_1 taken about the point x, y, z, as centre of gravity, and the other component being a relative-mean-motion, of which the mean components of momentum taken over the space S_1 everywhere vanish, it follows:—

(1) *That the resulting equations of motion must contain an expression for the rate of transformation from energy of mean-mean-motion to energy of relative-mean-motion, as well as the expressions for the transformation of the respective energies of mean- and relative-mean-motion to energy of heat-motion.*

(2) *That, when integrated over a complete system these equations must show that the possibility of the maintenance of the energy of relative-mean-motion depends, whatsoever may be the conditions, on the possible order of magnitudes of the periods of the relative-mean-motion, as compared with the periods of the heat-motions.*

The Equations of Mean- and Relative-mean-motion.

15. These last conclusions, besides bringing the general results of the previous argument to the test point, suggest the manner of adaptation of the equations of motion, by which the test may be applied.

Put $$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w' \qquad (11),$$

where $$\bar{u} = \frac{\Sigma(\rho u)}{\Sigma(\rho)}, \text{ \&c., \&c.} \qquad (12),$$

the summation extending over the space S_1 of which the centre of gravity is at the point x, y, z. Then since u, v, w are continuous functions of x, y, z, therefore $\bar{u}$, $\bar{v}$, $\bar{w}$, and u', v', w', are continuous functions of x, y, z. And as ρ is assumed constant, the equations of continuity for the two systems of motion are:

$$\frac{d\bar{u}}{dx}+\frac{d\bar{v}}{dy}+\frac{d\bar{w}}{dz}=0 \quad \text{and} \quad \frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}=0 \ldots\ldots\ldots\ldots\ldots(13);$$

also both systems of motions must satisfy the boundary conditions, whatever they may be.

Further putting $\bar{p}_{xx}$, &c., for the mean values of the stresses taken over the space S_1 and

$$p'_{xx}=p_{xx}-\bar{p}_{xx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

and defining S_1 to be such that the space variations of $\bar{u}, \bar{v}, \bar{w}$ are approximately constant over this space, we have, putting $\overline{u'u'}$, &c., for the mean values of the squares and products of the components of relative-mean-motion, for the equations of mean-mean-motion,

$$\left.\begin{aligned} \rho\frac{d\bar{u}}{dt} &= -\left\{\frac{d}{dx}\left(\bar{p}_{xx}+\rho\overline{uu}+\rho\overline{u'u'}\right)\right. \\ &\quad +\frac{d}{dy}\left(\bar{p}_{yx}+\rho\overline{uv}+\rho\overline{u'v'}\right) \\ &\quad \left.+\frac{d}{dz}\left(\bar{p}_{zx}+\rho\overline{uw}+\rho\overline{u'w'}\right)\right\} \\ \&c. &= \qquad \&c. \\ \&c. &= \qquad \&c. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

which equations are approximately true at every point in the same sense as that in which the equations (1) of mean-motion are true.

Subtracting these equations of mean-mean-motion from the equations of mean-motion, we have

$$\rho\frac{du'}{dt}=-\left\{\begin{aligned} &\frac{d}{dx}\left\{p'_{xx}+\rho\left(\bar{u}u'+u'\bar{u}\right)+\rho\left(u'u'-\overline{u'u'}\right)\right\} \\ &+\frac{d}{dy}\left\{p'_{yx}+\rho\left(\bar{u}v'+u'\bar{v}\right)+\rho\left(u'v'-\overline{u'v'}\right)\right\} \\ &+\frac{d}{dz}\left\{p'_{zx}+\rho\left(\bar{u}w'+u'\bar{w}\right)+\rho\left(u'w'-\overline{u'w'}\right)\right\} \end{aligned}\right\} \&c., \&c.\ldots(16),$$

which are the equations of momentum of relative-mean-motion at each point.

Again, multiplying the equations of mean-mean-motion by $\bar{u}$, $\bar{v}$, $\bar{w}$ respectively, adding and putting $2\bar{E}=\rho\left(u^2+v^2+\bar{w}^2\right)$, we obtain

$$\left(\frac{d}{dt}+\bar{u}\frac{d}{dx}+\bar{v}\frac{d}{dy}+\bar{w}\frac{d}{dz}\right)\bar{E}$$

$$=-\left\{\begin{array}{l}\frac{d}{dx}[\bar{u}(\bar{p}_{xx}+\rho\overline{u'u'})]+\frac{d}{dy}[\bar{u}(\bar{p}_{yx}+\rho\overline{u'v'})]+\frac{d}{dz}[\bar{u}(\bar{p}_{zx}+\rho\overline{u'w'})]\\+\frac{d}{dx}[\bar{v}(\bar{p}_{xy}+\rho\overline{v'u'})]+\frac{d}{dy}[\bar{v}(\bar{p}_{yy}+\rho\overline{v'v'})]+\frac{d}{dz}[\bar{v}(\bar{p}_{zy}+\rho\overline{v'w'})]\\+\frac{d}{dx}[\bar{w}(\bar{p}_{xz}+\rho\overline{w'u'})]+\frac{d}{dy}[\bar{w}(\bar{p}_{yz}+\rho\overline{w'v'})]+\frac{d}{dz}[\bar{w}(\bar{p}_{zz}+\rho\overline{w'w'})]\end{array}\right\}$$

$$+\left\{\begin{array}{l}\bar{p}_{xx}\frac{d\bar{u}}{dx}+\bar{p}_{yx}\frac{d\bar{u}}{dy}+\bar{p}_{zx}\frac{d\bar{u}}{dz}\\+\bar{p}_{xy}\frac{d\bar{v}}{dx}+\bar{p}_{yy}\frac{d\bar{v}}{dy}+\bar{p}_{zy}\frac{d\bar{v}}{dz}\\+\bar{p}_{xz}\frac{d\bar{w}}{dx}+\bar{p}_{yz}\frac{d\bar{w}}{dy}+\bar{p}_{zz}\frac{d\bar{w}}{dz}\end{array}\right\}+\rho\left\{\begin{array}{l}\overline{u'u'}\frac{d\bar{u}}{dx}+\overline{u'v'}\frac{d\bar{u}}{dy}+\overline{u'w'}\frac{d\bar{u}}{dz}\\+\overline{v'u'}\frac{d\bar{v}}{dx}+\overline{v'v'}\frac{d\bar{v}}{dy}+\overline{v'w'}\frac{d\bar{v}}{dz}\\+\overline{w'u'}\frac{d\bar{w}}{dx}+\overline{w'v'}\frac{d\bar{w}}{dy}+\overline{w'w'}\frac{d\bar{w}}{dz}\end{array}\right\}\quad\ldots(17),$$

which is the approximate equation of energy of mean-mean-motion in the same sense as the equation (3) of energy of mean-motion is approximate.

In a similar manner multiplying the equations (16) for the momentum of relative-mean-motion respectively by u', v', w', and adding, the result would be the equation for energy of relative-mean-motion at a point, but this would include terms of which the mean values taken over the space S_1 are zero, and, since all corresponding terms in the energy of heat are excluded, by summation over the space S_0 in the expression for the rate at which mean-motion is transformed into heat, there is no reason to include them for the space S_1; so that, omitting all such terms and putting

$$2\bar{E}'=\rho\,(\overline{u'^2}+\overline{v'^2}+\overline{w'^2})\quad\ldots\ldots\ldots\ldots(18),$$

we obtain

$$\left(\frac{d}{dt}+\bar{u}\frac{d}{dx}+\bar{v}\frac{d}{dy}+\bar{w}\frac{d}{dz}\right)\bar{E}'$$

$$=-\left\{\begin{array}{l}\frac{d}{dx}[u'(p'_{xx}+\rho u'u')]+\frac{d}{dy}[u'(p'_{yx}+\rho u'v')]+\frac{d}{dz}[u'(p'_{zx}+\rho u'w')]\\+\frac{d}{dx}[v'(p'_{yx}+\rho v'u')]+\frac{d}{dy}[v'(p'_{yy}+\rho v'v')]+\frac{d}{dz}[v'(p'_{zy}+\rho v'w')]\\+\frac{d}{dx}[w'(p'_{xz}+\rho w'u')]+\frac{d}{dy}[w'(p'_{yz}+\rho w'v')]+\frac{d}{dz}[w'(p'_{zz}+\rho w'w')]\end{array}\right\}$$

$$+\left\{\begin{array}{l}p'_{xx}\frac{du'}{dx}+p'_{yx}\frac{du'}{dy}+p'_{zx}\frac{du'}{dz}\\+p'_{xy}\frac{dv'}{dx}+p'_{yy}\frac{dv'}{dy}+p'_{zy}\frac{dv'}{dz}\\+p'_{xz}\frac{dw'}{dx}+p'_{yz}\frac{dw'}{dy}+p'_{zz}\frac{dw'}{dz}\end{array}\right\}-\left\{\begin{array}{l}\rho\overline{u'u'}\frac{d\bar{u}}{dx}+\rho\overline{u'v'}\frac{d\bar{u}}{dy}+\rho\overline{u'w'}\frac{d\bar{u}}{dz}\\+\rho\overline{v'u'}\frac{d\bar{v}}{dx}+\rho\overline{v'v'}\frac{d\bar{v}}{dy}+\rho\overline{v'w'}\frac{d\bar{v}}{dz}\\+\rho\overline{w'u'}\frac{d\bar{w}}{dx}+\rho\overline{w'v'}\frac{d\bar{w}}{dy}+\rho\overline{w'w'}\frac{d\bar{w}}{dz}\end{array}\right\}$$

$$\ldots\ldots\ldots\ldots(19),$$

where only the mean values, over the space S_1, of the expressions in the right member are taken into account.

This is the equation for the mean rate, over the space S_1, of change in the energy of relative-mean-motion per unit of volume.

It may be noticed that the rate of change in the energy of mean-mean-motion, together with the mean rate of change in the energy of relative-mean-motion, must be the total mean-rate of change in the energy of mean-motion, and that by adding the equations (17) and (19) the result is the same as is obtained from the equation (3) of energy of mean-motion by omitting all terms which have no mean value as summed over the space S_1.

The Expressions for Transformation of Energy from Mean-mean-motion to Relative-mean-motion.

16. When equations (17) and (19) are added together, the only expressions that do not appear in the equation of mean-energy of mean-motion are the last terms on the right of each of the equations, which are identical in form and opposite in sign.

These terms, which thus represent no change in the total energy of mean-motion, can only represent a transformation from energy of mean-mean-motion to energy of relative-mean-motion. And as they are the only expressions which do not form part of the general expression for the rate of change of the mean energy of mean-motion, they represent the total exchange of energy between the mean-mean-motion and the relative-mean-motion.

It is also seen that the action, of which these terms express the effect, is purely kinematical, depending simply on the instantaneous characters of the mean- and relative-mean-motion, whatever may be the properties of the matter involved, or the mechanical actions which have taken part in determining these characters. The terms, therefore, express the entire result of transformation from energy of mean-mean-motion to energy of relative-mean-motion, and of nothing but the transformation. Their existence thus completely verifies the first of the general conclusions in Art. 14.

The term last but one in the right member of the equation (17) for energy of mean-mean-motion, expresses the rate of transformation of energy of heat-motions to that of energy of mean-mean-motion, and is entirely independent of the relative-mean-motion.

In the same way, the term, last but one on the right of the equation (19) for energy of relative-mean-motion, expresses the rate of transformation from energy of heat-motions to energy of relative-mean-motion, and is quite independent of the mean-mean-motion.

17. In both equations (17) and (19) the first terms on the right express the rates at which the respective energies of mean- and relative-mean-motion are increasing on account of work done by the stresses on the mean- and relative-motions respectively, and by the additions of momentum caused by convections of relative-mean-motion by relative-mean-motion to the mean- and relative-mean-motions respectively.

It may also be noticed that while the first term on the right, in the equation (19) of energy of relative-mean-motion, is independent of mean-mean-motion, the corresponding term in equation (17) for mean-mean-motion is not independent of relative-mean-motion.

A Discriminating Equation.

18. In integrating the equations over a space moving with the mean-mean-motion of the fluid, the first terms on the right may be expressed as surface integrals, which integrals respectively express the rates at which work is being done on, and energy is being received across the surface, by the mean-mean-motion, and by the relative-mean-motion.

If the space over which the integration extends includes the whole system, or such part that the total energy conveyed across the surface by the relative-mean-motion is zero, then the rate of change in the total energy of relative-mean-motion within the space, is the difference of the integral, over the space, of the rate of increase of this energy by transformation from energy of mean-mean-motion, less the integral rate at which energy of relative-mean-motion is being converted into heat, or, integrating equation (19),

$$\iiint\left(\frac{d}{dt}+\bar{u}\frac{d}{dx}+\bar{v}\frac{d}{dy}+\bar{w}\frac{d}{dz}\right)\bar{E}'dx\,dy\,dz$$

$$=-\iiint\left\{\begin{matrix}\rho\overline{u'u'}\dfrac{d\bar{u}}{dx}+\rho\overline{u'v'}\dfrac{d\bar{u}}{dy}+\rho\overline{u'w'}\dfrac{d\bar{u}}{dz}\\+\rho\overline{v'u'}\dfrac{d\bar{v}}{dx}+\rho\overline{v'v'}\dfrac{d\bar{v}}{dy}+\rho\overline{v'w'}\dfrac{d\bar{v}}{dz}\\+\rho\overline{w'u'}\dfrac{d\bar{w}}{dx}+\rho\overline{w'v'}\dfrac{d\bar{w}}{dy}+\rho\overline{w'w'}\dfrac{d\bar{w}}{dz}\end{matrix}\right\}dx\,dy\,dz$$

$$+\iiint\left\{\begin{matrix}p'_{xx}\dfrac{du'}{dx}+p'_{yx}\dfrac{du'}{dy}+p'_{zx}\dfrac{du'}{dz}\\p'_{xy}\dfrac{dv'}{dx}+p'_{yy}\dfrac{dv'}{dy}+p'_{zy}\dfrac{dv'}{dz}\\p'_{xz}\dfrac{dw'}{dx}+p'_{yz}\dfrac{dw'}{dy}+p'_{zz}\dfrac{dw'}{dz}\end{matrix}\right\}dx\,dy\,dz \quad\text{..........(20).}$$

This equation expresses the fundamental relations:—

(1) *That the only integral effect of the mean-mean-motion on the relative-mean-motion is the integral of the rate of transformation from energy of mean-mean-motion to energy of relative-mean-motion.*

(2) *That, unless relative energy is altered by actions across the surface within which the integration extends, the integral energy of relative-mean-motion will be increasing, or diminishing, according as the integral rate of transformation from mean-mean-motion to relative-mean-motion is greater, or less than, the rate of conversion of the energy of relative-mean-motion into heat.*

19. For p'_{xx}, &c., are substituted their values as determined according to the theory of viscosity, the approximate truth of which has been verified, as already explained.

Putting

$$\left.\begin{aligned} p'_{xx} &= p + \tfrac{2}{3}\mu\left(\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz}\right) - 2\mu\frac{du}{dx}\text{, \&c., \&c.} \\ p'_{yx} &= -\mu\left(\frac{du'}{dy} + \frac{dv'}{dx}\right)\text{ \&c., \&c.} \end{aligned}\right\}\text{........(21),}$$

we have, substituting in the last term of equation (20), as the expression for the rate of conversion of energy of relative-mean-motion into heat,

$$\begin{aligned} -\iiint\frac{d}{dt}(pH)\,dx\,dy\,dz = \iiint\Bigg[p\left(\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz}\right) \\ -\mu\Bigg\{-\tfrac{2}{3}\left(\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz}\right)^2 + 2\left[\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2\right] \\ + \left(\frac{dw'}{dy} + \frac{dv'}{dz}\right)^2 + \left(\frac{du'}{dz} + \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dy}\right)^2\Bigg\}\Bigg]\,dx\,dy\,dz\text{ ......(22),} \end{aligned}$$

in which μ is a function of temperature only; or since ρ is here considered as constant,

$$\begin{aligned} -\iiint\frac{d}{dt}(pH')\,dx\,dy\,dz = -\mu\iiint\Bigg\{2\left[\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2\right] + \left(\frac{dw'}{dy} + \frac{dv'}{dx}\right)^2 \\ + \left(\frac{du'}{dz} + \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dy}\right)^2\Bigg\}\,dx\,dy\,dz\text{...(23),} \end{aligned}$$

whence substituting for the last term in equation (20) we have, if the energy of relative-mean-motion is maintained, neither increasing nor diminishing,

$$-\rho\iiint\left\{\begin{aligned} &\overline{u'u'}\,\frac{d\overline{u}}{dx} + \overline{u'v'}\,\frac{d\overline{u}}{dy} + \overline{u'w'}\,\frac{d\overline{u}}{dz} \\ &\quad + \overline{v'u'}\,\frac{d\overline{v}}{dx} + \overline{v'v'}\,\frac{d\overline{v}}{dy} + \overline{v'w'}\,\frac{d\overline{v}}{dz} \\ &\quad + \overline{w'u'}\,\frac{d\overline{w}}{dx} + \overline{w'v'}\,\frac{d\overline{w}}{dy} + \overline{w'w'}\,\frac{d\overline{w}}{dz} \end{aligned}\right\}dx\,dy\,dz$$

$$-\mu\iiint\left\{\begin{matrix} 2\left[\left(\frac{du'}{dx}\right)^2+\left(\frac{dv'}{dy}\right)^2+\left(\frac{dw'}{dz}\right)^2\right] \\ +\left(\frac{dw'}{dy}+\frac{dv'}{dx}\right)^2+\left(\frac{du'}{dz}+\frac{dw'}{dx}\right)^2 \\ +\left(\frac{dv'}{dx}+\frac{du'}{dy}\right)^2 \end{matrix}\right\} dx\,dy\,dz = 0 \;\ldots (24),$$

which is a discriminating equation as to the conditions under which relative-mean-motion can be sustained.

20. Since this equation is homogeneous in respect to the component velocities of the relative-mean-motion, it at once appears that it is independent of the energy of relative-mean-motion divided by the ρ. So that if μ/ρ is constant, the condition it expresses depends only on the relation between variations of the mean-mean-motion and the directional, or angular, distribution of the relative-mean-motion, and on the squares and products of the space periods of the relative-mean-motion.

And since the second term expressing the rate of conversion of heat into energy of relative-mean-motion is always negative, it is seen at once that, whatsoever may be the distribution and angular distribution of the relative-mean-motion and the variations of the mean-mean-motion, this equation must give an inferior limit for the rates of variation of the components of mean-mean-motion, in terms of the limits to the periods of relative-mean-motion, and μ/ρ, within which the maintenance of relative-mean-motion is impossible. And that, so long as the limits to the periods of relative-mean-motion are not infinite, this inferior limit to the rates of variation of the mean-mean-motion will be greater than zero.

Thus the second conclusion of Art. 14, and the whole of the previous argument is verified, and the properties of matter which prevent the maintenance of mean-motion, with periods of the same order of magnitude as those of the heat-motion, are shown to be amongst those properties of matter which are included in the equations of motion of which the truth has been verified by experience.

The Cause of Transformation.

21. The transformation function, which appears in the equations of mean-energy of mean- and relative-mean-motion, does not indicate the cause of transformation, but only expresses a kinematical principle as to the effect of the variations of mean-mean-motion, and the distribution of relative-mean-motion. In order to determine the properties of matter and the mechanical principles on which the effect of the variations of the mean-mean-motion on the distribution and angular distribution of relative-mean-motion depends, it is necessary to go back to the equations (16) of relative-

momentum at a point; and even then the cause is only to be found by considering the effects of the actions which these equations express in detail. The determination of this cause, though it in no way affects the proofs of the existence of the criterion as deduced from the equations, may be the means of explaining what has been hitherto obscure in the connection between thermodynamics and the principles of mechanics. That such may be the case, is suggested by the recognition of the separate equations of mean- and relative-mean-motion of matter.

The Equation of Energy of Relative-mean-motion and the Equation of Thermodynamics.

22. On consideration, it will at once be seen that there is more than an accidental correspondence between the equations of energy of mean- and relative-mean-motion respectively, and the respective equations of energy of mean-motion and of heat in thermodynamics.

If instead of including only the effects of the heat-motion on the mean-momentum, as expressed by p_{xx}, &c., the effects of relative-mean-motion are also included by putting p_{xx} for $\bar{p}_{xx} + \overline{\rho u' u'}$, &c., and p_{yz} for $\bar{p}_{yz} + \overline{\rho w' v'}$, &c., in equations (15) and (17), the equations (15) of mean-mean-motion become identical in form with the equations (1) of mean-motion, and the equation (17) of energy of mean-mean-motion becomes identical in form with the equation (3) of energy of mean-motion.

These equations, obtained from (15) and (17), being equally true with equations (1) and (3), the mean-mean-motion in the former being taken over the space S_1 instead of S_0 as in the latter, then, instead of equation (9), we should have for the value of the last term—

$$p_{xx} \frac{d\bar{u}}{dx} + \&c., = -\frac{d(pH)}{dt} + \overline{\rho u' u'} \frac{d\bar{u}}{dx} + \&c. \quad \ldots\ldots\ldots\ldots\ldots(25),$$

in which the right member expresses the rate at which heat is converted into energy of mean-mean-motion, together with the rate at which energy of relative-mean-motion is transformed into energy of mean-mean-motion; while equation (19) shows whence the transformed energy is derived.

The similarity of the parts taken by the transformation of mean-mean-motion into relative-mean-motion, and the conversion of mean-motion into heat, indicates that these parts are identical in form; or that the conversion of mean-motion into heat is the result of transformation, and is expressible by a transformation function similar in form to that for relative-mean-motion, but in which the components of relative-motion are the components of the heat-motions, and the density is the actual density at each point. Whence it would appear that the general equations, of which equations (19) and (16) are respectively the adaptations to the special condition of uniform density, must, by indicating the properties of matter involved, afford mechanical explanations of the law of universal dissipation of energy and of the second law of thermodynamics.

The proof of the existence of a criterion, as obtained from the equations, is quite independent of the properties and mechanical principles on which the effect of the variations of mean-mean-motion on the distribution of relative-mean-motion depends. And as the study of these properties and principles requires the inclusion of conditions which are not included in the equations of mean-motion of incompressible fluid, it does not come within the purpose of this paper. It is therefore reserved for separate investigation by a more general method.

The Criterion of Steady Mean-motion.

23. As already pointed out, it appears from the discriminating equation that the possibility of the maintenance of a state of relative-mean-motion depends on μ/ρ, the variation of mean-mean-motion, and the periods of the relative-mean-motion.

Thus, if the mean-mean-motion is in direction x only, and varies in direction y only, if u', v', w' are periodic in directions x, y, z, a being the largest period in space, so that their integrals over a distance a in direction x are zero, and if the co-efficients of all the periodic factors are α, then putting

$$\pm\frac{d\bar{u}}{dy} = C_1^2,$$

and taking the integrals, over the space a^3, of the 18 squares and products in the last term on the left of the discriminating equation (24) to be

$$-18\mu C_2\left(\frac{2\pi}{a}\right)^2\alpha^2a^3,$$

the integral of the first term over the same space cannot be greater than

$$\rho C_2\alpha^2C_1^2a^3.$$

Then, by the discriminating equation, if the mean-energy of relative-mean-motion is to be maintained,

$$\rho C_1^2 \text{ is greater than } 700\,.\frac{\mu}{a^2},$$

or
$$\frac{\rho a^2}{\mu}\sqrt{\left(\frac{d\bar{u}}{dy}\right)^2} = 700 \quad\text{..........................(26)}$$

is a condition under which relative-mean-motion cannot be maintained in a fluid, of which the mean-mean-motion is constant in the direction of mean-mean-motion, and subject to a uniform variation at right angles to the direction of mean-mean-motion. It is not the actual limit, to obtain which it would be necessary to determine the actual forms of the periodic function for u', v', w', which would satisfy the equations of motion (15), (16), as well as the equation of continuity (13), and to do this the functions would be of the form

$$\Sigma\left[A_r \cos\left\{r\left(nt+\frac{2\pi}{a}x\right)\right\}\right],$$

where r has the values 1, 2, 3, &c. It may be shown, however, that the retention of the terms in the periodic series in which r is greater than unity would increase the numerical value of the limit.

24. It thus appears that the existence of the condition (26) within which no relative-mean-motion, completely periodic in the distance a, can be maintained, is a proof of the existence, for the same variation of mean-mean-motion, of an actual limit of which the numerical value is between 700 and infinity.

In viscous fluids, experience shows that the further kinematical conditions imposed by the equations of motion do not prevent such relative-mean-motion. Hence for such fluids equation (26) proves that the actual limit, which discriminates between the possibility and impossibility of relative-mean-motion completely periodic in a space a, is greater than 700.

Putting equation (26) in the form

$$\sqrt{\left(\frac{d\bar{u}}{dy}\right)^2}=700\,\frac{\mu}{\rho a^2},$$

it at once appears that this condition does not furnish a criterion as to the possibility of the maintenance of relative-mean-motion, irrespective of its periods, for a certain condition of variation of mean-mean-motion. For by taking a^2 large enough, such relative-mean-motion would be rendered possible whatever might be the variation of the mean-mean-motion.

The existence of a criterion is thus seen to depend on the existence of certain restrictions to the value of the periods of relative-mean-motion—on the existence of conditions which impose superior limits on the values of a.

Such limits to the maximum values of a may arise from various causes. If $d\bar{u}/dy$ is periodic, the period would impose such a limit, but the only restrictions which it is my purpose to consider in this paper, are those which arise from the solid surfaces between which the fluid flows. These restrictions are of two kinds—restrictions to the motions normal to the surfaces, and restrictions tangential to the surfaces—the former are easily defined, the latter depend for their definition on the evidence to be obtained from experiments such as those of Poiseuille, and I shall proceed to show that these restrictions impose a limit to the value of a, which is proportional to D, the dimension between the surfaces. In which case, if

$$\sqrt{\overline{\left(\frac{du}{dy}\right)^2}} = \frac{U}{D},$$

equation (26) affords a proof of the existence of a criterion

$$\rho \cdot \frac{DU}{\mu} = K \quad \text{.................................(27)}$$

of the conditions of mean-mean-motion under which relative or sinuous-motion can continuously exist in the case of a viscous fluid between two continuous surfaces perpendicular to the direction y, one of which is maintained at rest, and the other in uniform tangential-motion in the direction x with velocity U.

SECTION III.

The Criterion of the Conditions under which Relative-mean-motion cannot be maintained in the case of Incompressible Fluid in Uniform Symmetrical Mean-flow between Parallel Solid Surfaces.—Expression for the Resistance.

25. The only conditions, under which definite experimental evidence as to the value of the criterion has as yet been obtained, are those of steady flow through a straight round tube of uniform bore; and for this reason it would seem desirable to choose for theoretical application the case of a round tube. But inasmuch as the application of the theory is only carried to the point of affording a proof of the existence of an inferior limit to the value of the criterion, which shall be greater than a certain quantity determined by the density and viscosity of the fluid and the conditions of flow, and as the necessary expressions for the round tube are much more complex than those for parallel plane surfaces, the conditions here considered are those defined by such surfaces.

Case I. Conditions.

26. The fluid is of constant density ρ and viscosity μ, and is caused to flow, by a uniform variation of pressure $d\bar{p}/dx$, in direction x between parallel surfaces, given by

$$y = -b_0, \quad y = b_0 \quad \text{..............................(28),}$$

the surfaces being of indefinite extent in directions z and x.

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The Boundary Conditions.

(1) There can be no motion normal to the solid surfaces, therefore

$$v = 0 \text{ when } y = \pm b_0 \quad \text{.........................(29).}$$

(2) That there shall be no tangential motion at the surface, therefore

$$u = w = 0 \text{ when } y = \pm b_0 \quad \text{.........................(30);}$$

whence by equation (21), putting u for u', $p_{yx} = -\mu du/dy$.

By the equation of continuity $du/dx + dv/dy + dw/dz = 0$, therefore at the boundaries we have the further conditions, that when $y = \pm b_0$,

$$du/dx = dv/dy = dw/dz = 0 \quad \text{.........................(31).}$$

Singular Solution.

27. If the mean-motion is everywhere in direction x, then, by the equation of continuity, it is constant in this direction, and as shown (Art. 8) the periods of mean-motion are infinite, and the equations (1), (3), and (9) are strictly true. Hence if

$$\bar{v} = \bar{w} = u' = v' = w' = 0 \quad \text{.........................(32),}$$

we have conditions under which a singular solution of the equations, applied to this case, is possible whatsoever may be the value of b_0, dp/dx, ρ and μ.

Substituting for p_{xx}, p_{yz}, &c., in equations (1) from equations (21), and substituting u for u', &c., these become

$$\rho \frac{du}{dt} = -\frac{dp}{dx} + \mu\left(\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) \quad \text{.........................(33).}$$

This equation does not admit of solution from a state of rest*; but assuming a condition of steady motion such that du/dt is everywhere zero, and dp/dx constant, the solution of

* In a paper on the "Equations of Motion and the Boundary Conditions of Viscous Fluid," read before Section A at the meeting of the B. A., 1883, I pointed out the significance of this disability to be integrated, as indicating the necessity of the retention of terms of higher orders to complete the equations, and advanced certain confirmatory evidence as deduced from the theory of gases. The paper was not published, as I hoped to be able to obtain evidence of a more definite character, such as that which is now adduced in Articles 7 and 8 of this paper, which shows that the equations are incomplete, except for steady motion, and that to render them integrable from rest the terms of higher orders must be retained, and thus confirms the argument I advanced, and completely explains the anomaly. (See Paper 46, page 132.)

$$\left.\begin{aligned} &\frac{\mu}{\rho}\left(\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)-\frac{1}{\rho}\frac{dp}{dx}=0, \\ \text{if} \qquad &u=du/dz=0 \text{ when } y=\pm b_0, \\ \text{is} \qquad &u=\frac{1}{\mu}\frac{dp}{dx}\frac{y^2-b_0^2}{2} \end{aligned}\right\}\quad\text{.................(34).}$$

This is a possible condition of steady motion, in which the periods of u, according to Art. 8, are infinite; so that the equations for mean-motion as affected by heat-motion, by Art. 8, are exact, whatever may be the values of

$$u, \ b_0, \ \rho, \ \mu, \text{ and } dp/dx.$$

The last of equations (34) is thus seen to be a singular solution of the equations (15) for steady mean-flow, or steady mean-mean-motion, when u', v', w', p', &c., have severally the values zero, and so the equations (16) of relative-mean-motion are identically satisfied.

In order to distinguish the singular values of u, I put

$$\left.\begin{aligned} &u=U, \qquad \int_{-b}^{b} u\,dy = 2b_0 U_m; \\ \text{whence} \qquad &\frac{dp}{dx}=-\frac{3\mu}{b_0^2}U_m, \quad U=\frac{3}{2}U_m\frac{b_0^2-y^2}{b_0^2} \end{aligned}\right\}\quad\text{.................(35).}$$

According to the equations, such a singular solution is always possible where the conditions can be realized, but the manner in which this solution of the equation (1) of mean-motion is obtained affords no indication as to whether or not it is the only solution—as to whether or not the conditions can be realized. This can only be ascertained either by comparing the results as given by such solutions with the results obtained by experiment, or by observing the manner of motion of the fluid, as in my experiments with colour bands.

The fact that these conditions are realized, under certain circumstances, has afforded the only means of verifying the truth of the assumptions as to the boundary conditions, that there shall be no slipping, and as to μ being independent of the variations of mean-motion.

Verification of the Assumptions in the Equation of Viscous Fluid.

28. As applied to the conditions of Poiseuille's experiments and similar experiments made since, the results obtained from the theory are found to agree throughout the entire range so long as u', v', w' are zero, showing that if there were any slipping it must have been less than the thousandth part of the mean-flow, although the tangential force at the boundary was 0·2 gr.

per square centimetre, or over 6 lbs. per square foot, the mean flow 376 millims. (1·23 feet) per second, and

$$d\bar{u}/dr = 215{,}000,$$

the diameter of this tube being 0·014 millim., the length 1·25 millims., and the head 30 inches of mercury.

Considering that the skin resistance of a steamer going at 25 knots is not 6 lbs. per square foot, it appears that the assumptions, as to the boundary conditions and the constancy of μ, have been verified under more exigent circumstances, both as regards tangential resistance and rate of variation of tangential stress, than occur in anything but exceptional cases.

Evidence that other Solutions are possible.

29. The fact that steady mean-motion is almost confined to capillary tubes, and that in larger tubes, except when the motion is almost insensibly slow, the mean-motion is sinuous and full of eddies, is abundant evidence of the possibility, under certain conditions, of solutions other than the singular solutions.

In such solutions u', v', w' have values, which are maintained, not as a system of steady periodic motion, but such as has a steady effect on the mean-flow through the tube; and equations (1) are only approximately true.

The Application of the Equations of the Mean- and Relative-mean-motion.

30. Since the components of mean-mean-motion in directions y and z are zero, and the mean flow is steady,

$$\bar{v} = 0, \quad \bar{w} = 0, \quad d\bar{u}/dt = 0, \quad d\bar{u}/dx = 0 \qquad\text{(36)},$$

and as the mean values of functions of u', v', w' are constant in the direction of flow,

$$\frac{d\,(\overline{u'u'})}{dx} = 0, \quad \frac{d\,(\overline{v'u'})}{dx} = 0, \quad \frac{d\,(\overline{w'u'})}{dx} = 0, \text{ \&c.} \qquad\text{(37)}.$$

By equations (21) and (37) the equations (15) of mean-motion become

$$\left.\begin{aligned}
\rho\frac{d\bar{u}}{dt} &= -\frac{dp}{dx} + \mu\left(\frac{d^2\bar{u}}{dy^2} + \frac{d^2\bar{u}}{dz^2}\right) - \rho\left\{\frac{d}{dy}(\overline{u'v'}) + \frac{d}{dz}(\overline{u'w'})\right\}\\
\rho\frac{d\bar{v}}{dt} &= -\frac{dp}{dy} \qquad\qquad\qquad\qquad - \rho\left\{\frac{d}{dy}(\overline{v'v'}) + \frac{d}{dz}(\overline{v'w'})\right\}\\
\rho\frac{d\bar{w}}{dt} &= -\frac{dp}{dz} \qquad\qquad\qquad\qquad - \rho\left\{\frac{d}{dy}(\overline{w'v'}) + \frac{d}{dz}(\overline{w'w'})\right\}
\end{aligned}\right\} \quad\text{(38)}.$$

The equation of energy of mean-mean-motion (17) becomes

$$\left.\begin{aligned}\frac{d(\bar{E})}{dt} = -\bar{u}\,\frac{d\bar{p}}{dx} + \mu\left\{\frac{d}{dy}\left(\bar{u}\,\frac{d\bar{u}}{dy}\right) + \frac{d}{dz}\left(\bar{u}\,\frac{d\bar{u}}{dz}\right)\right\} \\ -\rho\left\{\frac{d}{dy}(\bar{u}\,\overline{u'v'}) + \frac{d}{dz}(\bar{u}\,\overline{u'w'})\right\} - \mu\left\{\left(\frac{d\bar{u}}{dy}\right)^2 + \left(\frac{d\bar{u}}{dz}\right)^2\right\} \\ +\rho\left\{\overline{u'v'}\,\frac{d\bar{u}}{dy} + \overline{u'w'}\,\frac{d\bar{u}}{dz}\right\}\end{aligned}\right\} \quad \ldots\ldots(39).$$

Similarly the equation of mean-energy of relative-mean-motion (19) becomes

$$\begin{aligned}\frac{d\bar{E}'}{dt} = &-\frac{d}{dy}\left[u'(p'_{yx} + \rho\overline{u'v'}) + v'(p'_{yy} + \rho\overline{v'v'}) + w'(p'_{yz} + \rho\overline{w'v'})\right] \\ &-\frac{d}{dz}\left[u'(p'_{zx} + \rho\overline{u'w'}) + v'(p'_{zy} + \rho\overline{v'w'}) + w'(p_{zz} + \rho\overline{w'w'})\right] \\ &-\mu\left[2\left\{\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2\right\} + \left(\frac{dw'}{dy} + \frac{dv'}{dz}\right)^2\right. \\ &\qquad\left. + \left(\frac{du'}{dz} + \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dz}\right)^2\right] \\ &-\rho\left\{\overline{u'v'}\,\frac{d\bar{u}}{dy} + \overline{u'w'}\,\frac{d\bar{u}}{dz}\right\} \quad \ldots\ldots(40).\end{aligned}$$

Integrating in directions y and z between the boundaries and taking note of the boundary conditions by which $\bar{u}$, u', v', w' vanish at the boundaries together with the integrals, in direction z, of

$$\frac{d}{dz}\left(\bar{u}\,\frac{d\bar{u}}{dz}\right), \quad \frac{d}{dz}\left[\bar{u}'\,\rho\overline{u'w'}\right], \quad \frac{d}{dz}\left[u'(p_{zx} + \rho\overline{u'w'})\right], \text{ \&c.},$$

the integral equation of energy of mean-mean-motion becomes

$$\begin{aligned}\iint\frac{d\bar{E}}{dt}\,dy\,dz = -\iint\left[\bar{u}\,\frac{dp}{dx} + \mu\left\{\left(\frac{d\bar{u}}{dy}\right)^2 + \left(\frac{d\bar{u}}{dz}\right)^2\right\}\right. \\ \left. -\rho\left\{\overline{u'v'}\,\frac{d\bar{u}}{dy} + \overline{u'w'}\,\frac{d\bar{u}}{dz}\right\}\right] dy\,dz \quad \ldots\ldots(41).\end{aligned}$$

The integral equation of energy of relative-mean-motion becomes

$$\begin{aligned}\iint\frac{d\bar{E}'}{dt}\,dy\,dz = &-\iint\left[\rho\left\{\overline{u'v'}\,\frac{d\bar{u}}{dy} + \overline{u'w'}\,\frac{d\bar{u}}{dz}\right\}\right] dy\,dz \\ &-\mu\iint\left[2\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2\right. \\ &\left. + \left(\frac{dw'}{dy} + \frac{dv'}{dz}\right)^2 + \left(\frac{du'}{dz} + \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dy}\right)^2\right] dy\,dz \quad \ldots\ldots(42).\end{aligned}$$

If the mean-mean-motion is steady it appears from equation (41) that

$$-\iint \bar{u}\,\overline{\frac{dp}{dx}}\,dy\,dz,$$

the work done on the mean-mean-motion $\bar{u}$, per unit of length of the tube, by the constant variation of pressure, is in part transformed into energy of relative-mean-motion at a rate expressed by the transformation function:

$$-\iint \rho\left(\overline{u'v'}\,\overline{\frac{du}{dy}} + \overline{u'w'}\,\overline{\frac{dw}{dy}}\right) dy\,dz,$$

and in part transformed into heat at the rate:

$$\mu\iint\left[\left(\overline{\frac{du}{dy}}\right)^2 + \left(\overline{\frac{du}{dz}}\right)^2\right] dy\,dz.$$

While the equation (42) for the integral energy of relative-mean-motion shows that the only energy received by the relative-mean-motion is that transformed from mean-mean-motion, and the only energy lost by relative-mean-motion is that converted into heat by the relative-mean-motion at the rate expressed by the last term.

And hence if the integral of E' is maintained constant, the rate of transformation from energy of mean-mean-motion must be equal to the rate at which energy of relative-mean-motion is converted into heat, and the discriminating equation becomes

$$\iint \rho\left(\overline{u'v'}\,\overline{\frac{du}{dy}} + \overline{u'w'}\,\overline{\frac{du}{dz}}\right) dy\,dz = -\mu\iint\left[2\left\{\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2 + \left(\frac{dw'}{dz}\right)^2\right\} + \left(\frac{dw'}{dy} + \frac{dv'}{dz}\right)^2 + \left(\frac{du'}{dz} + \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} + \frac{du'}{dy}\right)^2\right] dy\,dz \quad \ldots\ldots(43).$$

The Conditions to be Satisfied by $\bar{u}$ and u', v', w'.

31. If the mean-mean-motion is steady $\bar{u}$ must satisfy :—

(1) The boundary conditions

$$\bar{u} = 0 \text{ when } y = \pm b_0 \quad \ldots\ldots\ldots\ldots(44);$$

(2) The equation of continuity

$$d\bar{u}/dx = 0 \quad \ldots\ldots\ldots\ldots(45);$$

(3) The first of the equations of motion (38)

$$\frac{dp}{dx} = \mu\left(\frac{d^2\bar{u}}{dy^2} + \frac{d^2\bar{u}}{dz^2}\right) - \rho\left\{\frac{d}{dy}(\overline{u'v'}) + \frac{d}{dz}(\overline{u'w'})\right\} \quad \ldots\ldots\ldots\ldots(46);$$

or putting $\bar{u} = U + \bar{u} - U$,

and $\dfrac{dp}{dx} = \mu \dfrac{d^2 U}{dy^2}$ as in the singular solution,

equation (46) becomes

$$\mu \left(\frac{d^2 (\bar{u} - U)}{dy^2} + \frac{d^2 (\bar{u} - U)}{dz^2} \right) = \rho \left\{ \frac{d}{dy} (\overline{u'v'}) + \frac{d}{dz} (\overline{u'w'}) \right\} \ldots\ldots (47);$$

(4) The integral of (47) over the section of which the left member is zero, and

$$\text{the mean value of } \mu d\bar{u}/dy = \mu dU/dy \text{ when } y = \pm b_0 \ldots\ldots (48).$$

From the condition (3) it follows that if $\bar{u}$ is to be symmetrical with respect to the boundary surfaces, the relative-mean-motion must extend throughout the tube, so that

$$\int_{-\infty}^{\infty} \left[\frac{d}{dy} (\overline{u'v'}) + \frac{d}{dz} (\overline{u'w'}) \right] dz \text{ is a function of } y^2 \ldots\ldots\ldots (49).$$

And as this condition is necessary, in order that the equations (38) of mean-mean-motion and the equations (16) of relative-mean-motion may be satisfied for steady mean-motion, it is assumed as one of the conditions for which the criterion is sought.

The components of relative-mean-motion must satisfy the periodic conditions as expressed in equations (12), which become, putting $2c$ for the limit in direction z,

(1)
$$\left.\begin{aligned} &\int_0^a u'\,dx = \int_0^a v'\,dx = \int_0^a w'\,dx = 0 \\ &\int_{-b_0}^{b_0} \int_{-c}^{c} u'\,dy\,dz = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (50).$$

(2) The equation of continuity

$$du'/dx + dv'/dy + dw'/dz = 0.$$

(3) The boundary conditions which with the equation of continuity give

$$u' = v' = w' = du'/dx = dv'/dy = dw'/dz = 0 \text{ when } y = \pm b_0 \ldots\ldots (51).$$

(4) The condition imposed by symmetrical mean-motion

$$\int_{-c}^{c} \left[\frac{d}{dy} (\overline{u'v'}) + \frac{d}{dz} (\overline{u'w'}) \right] dz = 2c \,.\, f(y^2) \ldots\ldots\ldots\ldots (52).$$

These conditions (1 to 4) must be satisfied, if the effect on $\bar{u}$ is to be symmetrical however arbitrarily u', v', w' may be superimposed on the mean-motion which results from a singular solution.

(5) If the mean-motion is to remain steady u', v', w' must also satisfy the kinematical conditions obtained by eliminating $\bar{p}$ from the equations of mean-mean-motion (38) and those obtained by eliminating p' from the equations of relative-mean-motion (16).

Conditions (1 to 4) determine an inferior Limit to the Criterion.

32. The determination of the kinematic conditions (5) is, however, practically impossible; but if they are satisfied, u', v', w' must satisfy the more general conditions imposed by the discriminating equation. From which it appears that when u', v', w' are such as satisfy the conditions (1 to 4), however small their values relative to $\bar{u}$ may be, if they be such that the rate of conversion of energy of relative-mean-motion into heat is greater than the rate of transformation of energy of mean-mean-motion into relative-mean-motion, the energy of relative-mean-motion must be diminishing. Whence, when u', v', w' are taken such periodic functions of x, y, z, as under conditions (1 to 4) render the value of the transformation function relative to the value of the conversion function a maximum, if this ratio is less than unity, the maintenance of any relative-mean-motion is impossible. And whatever further restrictions might be imposed by the kinematical conditions, the existence of an inferior limit to the criterion is proved.

Expressions for the Components of possible Relative-mean-motion.

33. To satisfy the first three of the equations (50) the expressions for u', v', w', must be continuous periodic functions of x, with a maximum periodic distance a, such as satisfy the conditions of continuity.

Putting

$$l = 2\pi/a\text{; and } n \text{ for any number from 1 to } \infty,$$

and

$$\left.\begin{aligned} u' &= \Sigma_0^\infty \left\{\left(\frac{d\alpha_n}{dy} + \frac{d\gamma_n}{dz}\right)\cos(nlx) + \left(\frac{d\beta_n}{dy} + \frac{d\delta_n}{dz}\right)\sin(nlx)\right\} \\ v' &= \Sigma_0^\infty \{nl\alpha_n \sin(nlx) - nl\beta_n \cos(nlx)\} \\ w' &= \Sigma_0^\infty \{nl\gamma_n \sin(nlx) - nl\delta_n \cos(nlx)\} \end{aligned}\right\} \quad \ldots\ldots(53),$$

u', v', w' satisfy the equation of continuity. And, if

$$\left.\begin{aligned} &\alpha = \beta = \gamma = \delta = d\alpha/dy = d\beta/dy = d\gamma/dz = d\delta/dz = 0 \text{ when } y = \pm b_0 \\ &\text{and } \alpha\beta,\ \alpha\gamma,\ \alpha\delta \text{ are all functions of } y^2 \text{ only,} \end{aligned}\right\} \ldots\ldots(54),$$

it would seem that the expressions are the most general possible for the components of relative-mean-motion.

Cylindrical-relative-motion.

34. If the relative-mean-motion, like the mean-mean-motion, is restricted to motion parallel to the plane of xy,

$$\gamma = \delta = w' = 0, \text{ everywhere} \quad \ldots\ldots\ldots\ldots\ldots\ldots(55),$$

and the equations (53) express the most general forms for u', v' in case of such cylindrical disturbance.

Such a restriction is perfectly arbitrary, and having regard to the kinematical restrictions, over and above those contained in the discriminating equation, would entirely change the character of the problem. But as no account of these extra kinematical restrictions is taken in determining the limit to the criterion, and as it appears from trial that the value found for this limit is essentially the same, whether the relative-mean-motion is general or cylindrical, I only give here the considerably simpler analyses for the cylindrical motion.

The functions of Transformation of Energy and Conversion to Heat for Cylindrical Motion.

35. Putting $\frac{d}{dt}({}_pH')$ for the rate at which energy of relative-mean-motion is converted to heat per unit of volume, expressed in the right-hand member of the discriminating equation (43),

$$\iiint \frac{d}{dt}({}_pH')\,dxdydz$$
$$= \mu \iiint \left[2\left\{\left(\frac{du'}{dx}\right)^2 + \left(\frac{dv'}{dy}\right)^2\right\} + \left(\frac{du'}{dy}\right)^2 + \left(\frac{dv'}{dx}\right)^2 + 2\frac{du'}{dy}\frac{dv'}{dx}\right] dxdydz \;\ldots\ldots(56).$$

Then substituting for the values of u', v', w' from equations (53), and integrating in direction x over $2\pi/l$, and omitting terms the integral of which, in direction y, vanishes by the boundary conditions,

$$\iint \frac{d}{dt}({}_pH')\,dydz = \frac{\mu}{2}\iint \Sigma\left\{(nl)^4(\alpha_n^2+\beta_n^2) + 2(nl)^2\left[\left(\frac{d\alpha_n}{dy}\right)^2 + \left(\frac{d\beta_n}{dy}\right)^2\right]\right.$$
$$\left. + \left(\frac{d^2\alpha_n}{dy^2}\right)^2 + \left(\frac{d^2\beta_n}{dy^2}\right)^2\right\} dydz \;\ldots\ldots(57).$$

In a similar manner, substituting for u', v', integrating, and omitting terms which vanish on integration, the rate of transformation of energy from mean-mean-motion, as expressed by the left member in the discriminating equation (43), becomes

$$\iint \rho \overline{u'v'} \frac{d\bar{u}}{dy} dy dz = \tfrac{1}{2} \iint \Sigma \left[nl \left(\alpha_n \frac{d\beta_n}{dy} - \beta_n \frac{d\alpha_n}{dy} \right) \frac{d\bar{u}}{dy} \right] dy dz \ldots (58).$$

And, since by Art. 31, conditions (3) equation (47),

$$\mu \frac{d^2}{dy^2} (\bar{u} - U) = \rho \frac{d}{dy} (\overline{u'v'}), \ldots\ldots (59),$$

integrating and remembering the boundary conditions,

$$\mu \frac{d}{dy} (\bar{u} - U) = \rho \overline{u'v'}, \quad \mu (\bar{u} - U) = \rho \int_{-b_0}^{y} \overline{u'v'} dy \ldots\ldots (60).$$

And since at the boundary $\bar{u} - U$ is zero,

$$\rho \int_{-b_0}^{b_0} (u'v') dy = 0 \ldots\ldots (61).$$

Whence, putting $U + \bar{u} - U$ for $\bar{u}$ in the right member of equation (58), substituting for $\bar{u} - U$ from (60), integrating by parts, and remembering that

$$\frac{d^2 U}{dy^2} = -3 \frac{U_m}{b_0^2}, \text{ which is constant} \ldots\ldots (62),$$

also that
$$u'v' = \tfrac{1}{2} \Sigma \left\{ nl \left(\alpha_n \frac{d\beta_n}{dy} - \beta_n \frac{d\alpha_n}{dy} \right) \right\} \ldots\ldots (63),$$

we have for the transformation function:

$$\int_{-b_0}^{b_0} \left(\rho \overline{u'v'} \frac{du}{dy} \right) dy = \Sigma \left[\tfrac{3}{2} \rho \frac{U_m}{b_0^2} \int_{-b_0}^{b_0} dy \int_{-b_0}^{y} nl \left(\alpha_n \frac{d\beta_n}{dy} - \beta_n \frac{d\alpha_n}{dy} \right) dy \right.$$
$$\left. + \frac{\rho^2}{4\mu} \int_{-b_0}^{b} (nl)^2 \left(\alpha_n \frac{d\beta_n}{dy} - \beta_n \frac{d\alpha_n}{dy} \right)^2 dy \right] \ldots\ldots (64).$$

If u', v' are indefinitely small, the last term, which is of the fourth degree, may be neglected.

Substituting in the discriminating equation (43) this may be put in the form

$$\frac{2\rho b_0 U_m}{\mu}$$
$$= \frac{2b_0^3 \int_{-b_0}^{b_0} \Sigma \left\{ n^4 l^4 (\alpha_n^2 + \beta_n^2) + 2n^2 l^2 \left[\left(\frac{d\alpha_n}{dy} \right)^2 + \left(\frac{d\beta_n}{dy} \right)^2 \right] + \left(\frac{d^2\alpha_n}{dy^2} \right)^2 + \left(\frac{d^2\beta_n}{dy^2} \right)^2 \right\} dy}{3 \int_{-b_0}^{b_0} dy \int_{-b_0}^{y} \Sigma \left\{ nl \left(\beta_n \frac{d\alpha_n}{dy} - \alpha_n \frac{d\beta_n}{dy} \right) \right\} dy}$$
$$\ldots\ldots (65).$$

Limits to the Periods.

36. As functions of y, the variations of α_n, β_n are subject to the restrictions imposed by the boundary conditions, and in consequence their periodic distances are subject to superior limits determined by $2b_0$, the distance between the fixed surfaces.

In direction x, however, there is no such direct connection between the value of b_0 and the limits to the periodic distance, as expressed by $2\pi/nl$. Such limits necessarily exist, and are related to the limits of α_n and β_n in consequence of the kinematical conditions necessary to satisfy the equations of motion for steady mean-mean-motion; these relations, however, cannot be exactly determined without obtaining a general solution of the equations.

But from the form of the discriminating equation (43) it appears that no such exact determination is necessary in order to prove the inferior limit to the criterion.

The boundaries impose the same limits on α_n, β_n whatever may be the value of nl; so that if the values of α_n, β_n be determined so that the value of

$$\frac{2\rho b_0 U_m}{\mu} \text{ is a minimum}$$

for every value of nl, the value of rl, which renders this minimum a minimum-minimum may then be determined, and so a limit found to which the value of the complete expression approaches, as the series in both numerator and denominator become more convergent for values of nl differing in both directions from rl.

Putting l, α, β for rl, α_r, β_r respectively, and putting for the limiting value to be found for the criterion

$$K_1 = \frac{2\rho b_0 U_m}{\mu} \quad \text{.........................(66)}$$

$$\frac{3}{2}K_1 = b_0^3 \frac{\int_{-b_0}^{b_0} \left\{ l^4(\alpha^2+\beta^2) + 2l^2\left[\left(\frac{d\alpha}{dy}\right)^2 + \left(\frac{d\beta}{dy}\right)^2\right] + \left(\frac{d^2\alpha}{dy}\right)^2 + \left(\frac{d^2\beta}{dy}\right)^2 \right\} dy}{l\int_{-b_0}^{b} dy \int_{-b_0}^{y}\left(\beta\frac{d\alpha}{dy} - \alpha\frac{d\beta}{dy}\right) dy} \quad \text{...(67)}$$

where α and β are such functions of y that K_1 is a minimum whatever the value of l, and l is so determined as to render K_1 a minimum-minimum.

Having regard to the boundary conditions, &c., and omitting all possible terms which increase the numerator without affecting the denominator, the most general form appears to be

$$\left.\begin{aligned} \alpha &= \Sigma_0^n\,[a_{2s+1}\sin(2s+1)p],\\ \beta &= \Sigma_0^n\,[b_{2t}\sin(2tp)],\\ \text{where}\quad p &= \pi y/2b_0 \end{aligned}\right\}\quad\text{..................(68).}$$

To satisfy the boundary conditions

$s = 2r$, when s is even, $\qquad s = 2r+1$, when s is odd.

$t = 2r+1$, when t is odd, $\qquad t = 2(r+1)$, when t is even.

$$\left.\begin{aligned} &\text{Since } \alpha = 0, \text{ when } p = \pm\tfrac{1}{2}\pi,\\ &\qquad \Sigma_0^\infty (a_{4r+1} - a_{4r+3}) = 0,\\ &\text{and since } d\beta/dy = 0, \text{ when } p = \pm\tfrac{1}{2}\pi,\\ &\qquad \Sigma_0^\infty \{-(4r+2)\,b_{4r+2} + 4(r+1)\,b_{4r+4}\} = 0 \end{aligned}\right\}\quad\text{..............(69).}$$

From the form of K_1 it is clear that every term in the series for α and β increases the value of K_1 and to an extent depending on the value of r. K_1 will therefore be a minimum, when

$$\left.\begin{aligned} \alpha &= a_1\sin p + a_3\sin 3p\\ \beta &= b_2\sin 2p + b_4\sin 4p, \end{aligned}\right\}\text{........................(70),}$$

which satisfy the boundary conditions if

$$\left.\begin{aligned} a_3 &= a_1\\ b_2 &= 2b_4 \end{aligned}\right\}\quad\text{..................................(71).}$$

Therefore we have, as the values of α and β, which render K_1 a minimum for any value of l

$$\left.\begin{aligned} &\alpha/a_1 = \sin p + \sin 3p, \quad \beta/b_2 = \sin 2p + \tfrac{1}{2}\sin 4p.\\ &\text{And}\\ &\frac{2b_0}{\pi a_1}\frac{d\alpha}{dy} = \cos p + 3\cos 3p, \quad \frac{2b_0}{\pi b_2}\frac{d\beta}{dy} = 2\cos 2p + 2\cos 4p\\ &\frac{2b_0}{\pi a_1 b_2}\left(\frac{\alpha d\beta}{dy} - \frac{\beta d\alpha}{dy}\right) = \frac{1}{4}\{-3\sin p - 3\sin 3p + \sin 5p + \sin 7p\} \end{aligned}\right\}\text{...(72)}$$

and integrating twice

$$l\int_{-b_0}^{b_0} dy \int_{-b_0}^{y}\left(\beta\frac{d\alpha}{dy} = \alpha\frac{d\beta}{dy}\right)dy = -1{\cdot}325\,l\,\frac{2b_0}{\pi}\,a_1 b_2\ \text{.........(73).}$$

Putting $\qquad \dfrac{\pi}{2b_0}L$ for l,

the denominator of $\dfrac{3}{2}K_1$, equation (67), becomes

$$-1{\cdot}325La_1b_2.$$

In a similar manner the numerator is found to be

$$b_0^4\left(\frac{\pi}{2b_0}\right)^4\{L^4(2a_1^2+1\cdot25b_2^2)+2L^2(10a_1^2+8b_2^2)+82a_1^2+80b_2^2\},$$

and as the coefficients of a_1 and b_2 are nearly equal in the numerator, no sensible error will be introduced by putting

$$b_2=-a_1,$$

then
$$\frac{3}{2}K_1=\frac{L^4+2\times5\cdot53L^2+50}{0\cdot408L}\left(\frac{\pi}{2}\right)^4 \quad\ldots\ldots\ldots\ldots\ldots\ldots(74)$$

which is a minimum if

$$L=1\cdot62 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(75)$$

and
$$K_1=517 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(76).$$

Hence, for a flat tube of unlimited breadth, the criterion

$$\rho 2b_0U_m/\mu \text{ is greater than } 517 \quad\ldots\ldots\ldots\ldots\ldots\ldots(77).$$

37. This value must be less than that of the criterion for similar circumstances. How much less it is impossible to determine theoretically without effecting a general solution of the equations; and, as far as I am aware, no experiments have been made in a flat tube. Nor can the experimental value 1900, which I obtained for the round tube, be taken as indicative of the value for a flat tube, except that, both theoretically and practically, the critical value of U_m is found to vary inversely as the hydraulic mean depth, which would indicate that, as the hydraulic mean depth in a flat tube is double that for a round tube, the criterion would be half the value, in which case the limit found for K_1 would be about $0\cdot61K$. This is sufficient to show that the absolute theoretical limit found is of the same order of magnitude as the experimental value; so that the latter verifies the theory, which, in its turn, affords an explanation of the observed facts.

The State of Steady Mean-motion above the Critical Value.

38. In order to arrive at the limit for the criterion it has been necessary to consider the smallest values of u', v', w', and the terms in the discriminating equation of the fourth degree have been neglected. This, however, is only necessary for the limit, and, preserving these higher terms, the discriminating equation affords an expression for the resistance in the case of steady mean-mean-motion.

The complete value of the function of transformation as given in equation (64) is

$$\int_{-b_0}^{b_0}\left(\overline{\rho u'v'}\frac{d\bar{u}}{dy}\right) = \Sigma\left[\rho\frac{3U_m}{2b_0^2}\int_{-b_0}^{b_0}dy\int_{-b_0}^{y}nl\left(\alpha_n\frac{d\beta_n}{dy}-\beta_n\frac{d\alpha_n}{dz}\right)dy\right.$$

$$\left.+\frac{\rho^2}{4\mu}\int_{-b_0}^{b_0}(nl)^2\left(\alpha_n\frac{d\beta_n}{dy}-\beta\frac{d\alpha}{dy}\right)^2 dy\right] \quad\ldots\ldots(77a).$$

Whence putting $U+\bar{u}-U$, for $\bar{u}$ in the left member of equation (77), and integrating by parts, remembering the conditions, this member becomes

$$\frac{3U_m}{b_0^2}\int_{-b_0}^{b_0}dy\int_{-b_0}^{y}\rho\overline{u'v'}\,dy+\frac{\rho^2}{\mu}\int_{-b_0}^{b_0}(u'v')^2\,dy \quad\ldots\ldots\ldots\ldots(78),$$

in which the first term corresponds with the first term in the right member of equation (64), which was all that was retained for the criterion, and the second term corresponds with the second term in equation (64), which was neglected.

Since by equation (35)

$$\frac{3U_m}{b_0^2}=-\frac{1}{\mu}\frac{dp}{dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(78a),$$

we have, substituting in the discriminating equation (43), either

$$-\frac{2}{3}\rho\frac{b_0^3}{\mu^2}\frac{dp}{dx}=\frac{2b_0^3}{3}\left.\frac{\left(\int\frac{d/dt\,(pH')\,dy}{\mu}+\frac{\rho^2}{\mu^2}\int_{-b_0}^{b_0}(\overline{u'v'})^2\,dy\right)}{-\int_{-b_0}^{b_0}dy\int_{-b_0}^{y}\overline{u'v'}\,dy}\right\} \quad\ldots\ldots(79),$$

or

$$\mu\frac{d^2\bar{u}}{dy^2}-\frac{dp}{dx}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(80).$$

Therefore, as long as $\frac{2}{3}\rho\frac{b_0^3}{\mu^2}\frac{dp}{dx}$

is of constant value, there is dynamical similarity under geometrically similar circumstances.

The equation (79) shows that,

$$\text{when } -\frac{2}{3}\rho\frac{b_0^3}{\mu^2}\frac{dp}{dx} \text{ is greater than } K,$$

$\overline{u'v'}$ must be finite, and such that the last term in the numerator limits the rate of transformation, and thus prevents further increase of $\overline{u'v'}$.

The last term in the numerator of equation (79) is of the order and degree

$$\rho^2 L^4 \alpha^4/\mu^2 \text{ as compared with } L^4\alpha^2,$$

the order and degree of $\frac{1}{\mu}\frac{d}{dt}(pH')$ the first term in the numerator.

It is thus easy to see how the limit comes in. It is also seen from equation (79) that, above the critical value, the law of resistance is very complex and difficult of interpretation, except in so far as showing that the resistance varies as a power of the velocity higher than the first.

Some specific features of atmospheric tubulence*

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The spectrum of atmospheric turbulence is very broad by comparison with spectra in wind tunnels. We introduce the notion of small-scale and large-scale turbulence. Small-scale turbulence consists of a set of disturbances, the scales of which do not exceed the distance to the wall and for which the hypothesis of three-dimensional isotropy is valid in a certain rough approximation. Large-scale turbulence is essentially anisotropic; the horizontal scale in the atmosphere is much larger than the vertical one, the latter being confined to a certain characteristic height H. The horizontal scale varies widely according to the external conditions and characteristics of the medium.

The mean value of the internal scale of turbulence

$$l_1 = (\nu^3/\epsilon)^{\frac{1}{4}} \tag{1}$$

is of the order of 1 cm for the atmosphere. The maximum horizontal scale of turbulent inhomogeneities is of the order of $L_0 = 2500$ km according to different estimates.† The range of the spectrum of atmospheric turbulence in the horizontal direction is

$$L_0/l_1 = 2{\cdot}5 \times 10^8$$

or 28 octaves.

It is well known that the ratio of the external turbulent scale to the internal one is proportional to the Reynolds number to the power 3/4. The fluctuating character of the statistical properties of the small-scale turbulence is due to a very large spectrum width of atmospheric disturbances. As practice shows, it is enough to average the results of the spectral analysis of fluctuations (empirical energy distribution) over the period of 10 min in order to obtain relatively stable spectra of turbulence fluctuations in the surface layer. Using Kolmogorov's theory one can try to approximate these distributions by a relation of the form

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}. \tag{2}$$

* *Editors' footnote.* This paper is a record of part of a lecture given by the author at the IUGG–IUTAM Symposium on Fundamental Problems in Turbulence and their Relation to Geophysics, at Marseilles in September 1961. The part reproduced here concerns the local structure of turbulence in general, and was related to a lecture given by A. N Kolmogorov at another symposium in the preceding week (see the following paper in this journal). The two lectures are being included in the published records of the proceedings of the respective Symposia, and are published here also in view of their interest to a large number of English-speaking readers.

† The first estimate of this kind was provided by Defant (1921) on the basis of horizontal mixing study. Modern research on the correlation function behaviour of meteorological elements at distances of the order of some hundred and thousand km give similar values. The external scale in this case is expressed by a rather distinct break in the correlation functions.

As practice shows, one succeeds in the majority of cases with a rather high degree of an accuracy (of the order of 5 %) (Gurvitch 1960). Successive measurements show that, although each measurement is in satisfactory agreement with a $(-5/3)$-power law in a certain range of scales, the intensity of turbulence varies from measurement to measurement, which may be explained by variance of the energy dissipation rate ϵ (the main parameter of the locally isotopic theory). These slow fluctuations of energy dissipation are due to change of the large-scale processes in the observation region, or 'weather' in a general sense. Similar slow macroscopic changes of energy dissipation must be observed at very large Reynolds numbers and they are actually observed in the atmosphere.

The comparatively small statistical stability of the parameter ϵ requires a more distinct determination of a statistical ensemble by which the averaging is made in the theory of locally-isotropic turbulence.*

Let M_1 and M_2 be two observation points located at a distance apart which exceeds the internal scale of turbulence, but is small by comparison with the distance to the boundary of the flow (this is the condition of applicability of a local treatment of turbulence).

Let us consider the rate of dissipation averaged over some volume of a standard form connected invariantly with the points M_1 and M_2. As an example of such a volume V_{M_1,M_2} it is reasonable to take the interior of the sphere with the observation points as poles. Then the average is

$$\tilde{\epsilon}(M_1, M_2) = \frac{1}{\frac{1}{6}\pi r^3}\int_{V_{M_1,M_2}} \epsilon\, dV. \tag{3}$$

Assume that in every measurement of turbulent fluctuations, in particular, in the measurement of an instantaneous difference of velocity $\Delta\mathbf{v} = \mathbf{v}(M_2) - \mathbf{v}(M_1)$, the value $\tilde{\epsilon}$ is recorded simultaneously. Let us now fix the observation points (and the distance r also) and then select only those occasions at which $\tilde{\epsilon}$ takes a certain given value. On repeating this operation many times (under some like external conditions) we obtain the statistical ensemble which could conditionally be called a 'pure' one depending upon the selection of the observation points and the value of $\tilde{\epsilon}$ as parameters.

Now let us compute for this ensemble two-point mean values of momentum introduced by Kolmogorov (1941*a*):

$$B_{nn} = \langle(\Delta\mathbf{v})_n^2\rangle, \quad B_{dd} = \langle(\Delta\mathbf{v})_d^2\rangle, \quad B_{ddd} = \langle(\Delta\mathbf{v})_d^3\rangle.$$

In accordance with the general principle of local similarity these values will depend only upon $\tilde{\epsilon}$, upon the distance r and the internal Reynolds number

$$\tilde{R}(r) = \tilde{\epsilon}^{\frac{1}{3}} r^{\frac{4}{3}}/\nu. \tag{4}$$

For the mean-square of the longitudinal difference of velocities we have the form

$$\langle(\Delta\mathbf{v})_d^2\rangle_{\tilde{\epsilon}} = C(\tilde{R})\,\tilde{\epsilon}^{\frac{2}{3}} r^{\frac{2}{3}}. \tag{5}$$

* In the study of turbulence in wind tunnels one usually takes averages over a certain ensemble corresponding to a time ensemble, and the averaging period is larger than the life-time of the largest eddies. In the case of atmospheric turbulence there arise specific difficulties; so that the life-time of the largest eddies in the atmosphere greatly exceeds, as a rule, the time of measuring turbulent characteristics.

The second self-similarity hypothesis of Kolmogorov at rather large Reynolds numbers means in this scheme that if the distances r are large enough, and $\tilde{\epsilon}$ is fixed, the function $C(\tilde{R})$ can be replaced by a constant

$$C_0 = \lim_{\tilde{R}\to\infty} C(\tilde{R}). \tag{6}$$

This assumption will be called a similarity hypothesis for a 'pure' régime.

In practice we always deal with some mixed ensemble in which the value $\tilde{\epsilon}$ varies in accordance with some general statistical law. A natural time ensemble, corresponding to continuous observations of a certain limited part of a turbulent flow for a long enough period of time should be considered as a 'mixed régime'.

Let us consider now a simplified scheme, for which natural means (expectations) of the values required can be computed. Assume that for any choice of the observation points M_1 and M_2 the value $\tilde{\epsilon}$ has a logarithmically normal distribution.* This means that $\tilde{\epsilon}$ can be written in the form

$$\tilde{\epsilon} = \epsilon_0 \, e^{\eta}, \tag{7}$$

where ϵ_0 is the 'mean geometrical' dissipation value, and η is a random variable obeying the Gaussian distribution law with parameters

$$\overline{\eta} = \langle\eta\rangle = 0, \quad \mathscr{D}(\eta) = \langle\eta^2\rangle = \beta. \tag{8}$$

The logarithmic dispersion β is the main dimensionless parameter characterizing the statistical distribution. For the logarithmic distribution it is not difficult to compute the moment of any order to be

$$\langle\tilde{\epsilon}^{\alpha}\rangle = \epsilon_0^{\alpha} \, e^{\frac{1}{2}\alpha^2\beta}. \tag{9}$$

In particular, if $\alpha = 1$, we have

$$\overline{\epsilon} = \epsilon_0 \, e^{\frac{1}{2}\beta}. \tag{10}$$

Note that in contrast to ϵ_0 and β, which depend upon the distance r, the value $\overline{\epsilon}$ does not depend upon the observation distance r; $\overline{\epsilon}$ is the main macroscopic characteristic, which can be obtained, for example, from general considerations of the energy balance of the system.† Further, we shall consider $\overline{\epsilon}$ as a certain given constant.

Now compute mean-square fluctuation of $\tilde{\epsilon}$;

$$\sigma_{\tilde{\epsilon}}^2 = \langle\tilde{\epsilon}^2\rangle - \overline{\epsilon}^2 = \epsilon_0^2(e^{2\beta} - e^{\beta}). \tag{11}$$

The logarithmic dispersion β can be directly expressed in terms of the variance coefficient $M = \sigma/\overline{\epsilon}$;

$$e^{\beta} = 1 + M^2, \tag{12}$$

and

$$\epsilon_0 = \overline{\epsilon}(1+M^2)^{-\frac{1}{2}}. \tag{13}$$

* This assumption is not very restrictive as an approximate hypothesis since the distribution of any essentially positive characteristic can be represented by a logarithmically Gaussian distribution with correct values of the first two moments (see also Kolmogorov 1941*b*).

† For the study of turbulent structure in a wide range of scales in the atmosphere (flow over an infinite horizontal plane) it is reasonable to average over the horizontal co-ordinates, as the general statistical régime of motion can be considered horizontally homogeneous.

The computation of $\langle\tilde{\epsilon}^{\frac{2}{3}}\rangle$ gives

$$\langle\tilde{\epsilon}^{\frac{2}{3}}\rangle = \epsilon_0^{\frac{2}{3}}\, e^{2\beta/9}, \tag{14}$$

or, on using (10),

$$\langle\tilde{\epsilon}^{\frac{2}{3}}\rangle = \bar{\epsilon}^{\frac{2}{3}}\, e^{-\beta/9}. \tag{15}$$

This result can also be written with the aid of (12) in the form

$$\langle\tilde{\epsilon}^{\frac{2}{3}}\rangle = \bar{\epsilon}^{\frac{2}{3}}(1+M^2)^{-\frac{1}{9}}. \tag{16}$$

On averaging Kolmogorov's equation (the '(2/3)-power law'), and taking into account the possible dependence of the variance coefficient M upon the distance r, we obtain for the mixed régime the result

$$\langle(\Delta \mathrm{v})_d^2\rangle = C_0\bar{\epsilon}^{\frac{2}{3}}\frac{r^{\frac{2}{3}}}{\{1+M^2(r)\}^{\frac{1}{9}}}. \tag{17}$$

To obtain the dependence of M on r some additional assumptions are required.

Proceeding from the fact that the dependence of a longitudinal structure function upon the variance coefficient is very weak (if it exists at all), the dispersion $\mathscr{D}(\tilde{\epsilon})$ can be computed in the first approximation by taking the ordinary (2/3)-power law for the structure function if $r \gg l_1$. The fourth-product moments necessary for computation of the correlation function of disspation can be determined in this case from the Millionshchikov hypothesis (introducing some reduction factor). Thus, the following representation of the correlation function of energy dissipation can be obtained

$$\langle\epsilon'_{M_1}\epsilon'_{M_2}\rangle = \begin{Bmatrix}\gamma\bar{\epsilon}^2 & r \ll l_1, \\ \gamma\bar{\epsilon}^2(l_1/r)^{\frac{8}{3}} & r \gg l_1,\end{Bmatrix} \tag{18}$$

where $\epsilon' = \epsilon - \bar{\epsilon}$, and γ and γ' are the numerical coefficients which can depend on the Reynolds number.

The dispersion of averaged dissipation $\mathscr{D}(\tilde{\epsilon})$ can be computed on the basis of this correlation function by known methods. The dependence upon the distance r and other parameters can be described by the following approximate relations:

$$\mathscr{D}(\tilde{\epsilon}) = \begin{Bmatrix}\gamma\bar{\epsilon}^2 & r \ll l_1, \\ C\gamma'\bar{\epsilon}^2(l_1/r)^{\frac{8}{3}} & r \gg l_1,\end{Bmatrix} \tag{19}$$

where C is a universal numerical constant.*

By accepting an approximate dependence of the variance coefficient of dissipation upon the distance (if $r \gg l_1$) we obtain the value of the longitudinal structure function in the second approximation with correction for dissipation variance.

On expressing the internal scale in terms of the external scale and the Reynolds number, the result obtained can be represented in the form

$$\langle(\Delta \mathrm{v}^2)_d\rangle = \frac{C_0\bar{\epsilon}^{\frac{2}{3}}r^{\frac{2}{3}}}{\left\{1+\dfrac{C\gamma'}{R^2}\left(\dfrac{L_0}{r}\right)^{\frac{8}{3}}\right\}^{\frac{1}{9}}}. \tag{20}$$

* The question of dispersion of dissipation is treated in more detail in the work of Golitsyn (1962).

The computation given above is, of course, of a tentative character. However, it shows that for the mixed régime (natural time ensemble) one can notice a very weak deviation from the (2/3)-power law if Kolmogorov's hypotheses are considered valid for the 'pure' ensemble describing the local régime of turbulence at a given dissipation value. Another alternative is that the 'mixed' régime is asymptotically described by the (2/3)-power law, in which case the local régime with definite dissipation cannot strictly satisfy the second Kolmogorov hypothesis (self-similarity with respect to internal Reynolds number).

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PRÉCISIONS SUR LA STRUCTURE LOCALE DE LA TURBULENCE DANS UN FLUIDE VISQUEUX AUX NOMBRES DE REYNOLDS ÉLEVÉS

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Dans le programme du Symposium faisant suite à notre Colloque entre l'exposé de A. M. OBUKHOV : « Some Specific Features of Atmospheric Turbulence ». Un des chapitres de cet exposé est consacré à la précision de la représentation concernant la structure locale de la turbulence. Il a un grand intérêt pour tous les spécialistes de la théorie statistique de la turbulence. A la demande de certains participants du Colloque, je voudrais présenter ici avec plus de détails l'origine de ces précisions et donner l'exposé des derniers résultats obtenus par A. M. OBUKHOV dans ce domaine.

Elaborées en 1939-1941 par moi [1, 2, 3] et par A. M. OBUKHOV [4, 5] les représentations concernant la structure locale de la turbulence aux nombres de Reynolds élevés ont eu comme base l'idée de RICHARDSON de l'existence dans l'écoulement turbulent de tourbillons de toutes échelles possibles,

$$l < r < L$$

entre « l'échelle externe » L et « l'échelle interne » l, et de l'existence d'un certain mécanisme uniforme de la transmission de l'énergie des tourbillons d'échelle plus grande vers ceux plus fins.

Ces représentations, auxquelles sont arrivés d'autres auteurs, ont obtenu une très large extension. Cependant bientôt après leur apparition, L. D. LANDAU avait remarqué qu'elles ne prennent pas en considération le fait qui découle de l'hypothèse que le mécanisme de la transmission de l'énergie des tourbillons plus gros vers les plus fins a un caractère fortuit et chaotique : avec l'augmentation du rapport $\frac{L}{l}$, la variation de la dissipation de l'énergie

$$\varepsilon = \frac{\nu}{2} \cdot \sum_{\alpha} \sum_{\beta} \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right)^2$$

doit augmenter d'une façon illimitée. Plus exactement, il est naturel d'admettre que la dispersion du logarithme de la valeur ε a, lorsque $\frac{L}{l} \gg 1$, une valeur asymptotique de la forme

$$\sigma^2_{\log \varepsilon} \sim A + k' \log \frac{L}{l} \tag{1}$$

où k' est une certaine constante universelle.

C'est tout dernièrement que A. M. Obukhov avait tracé le chemin pour spécifier les résultats [1, 5] en prenant en considération la remarque de L. D. Landau. La méthode s'appuie sur l'examen de la dissipation de

$$\varepsilon_r(x, t) = \frac{3}{4\pi r^3} \int_{|h| \leqslant r} \varepsilon(\mathbf{x} + \mathbf{h}, t)\, dh$$

prise en moyenne pour une sphère de rayon r, et en même temps s'appuie sur la supposition qu'avec un rapport $\frac{L}{l}$ élevé, le logarithme de la valeur $\varepsilon_r(\mathbf{x}, t)$ a une répartition normale. Il est donc tout à fait naturel de considérer que la dispersion du logarithme de $\varepsilon_r(\mathbf{x}, t)$ a la forme

$$\sigma_r^2(\mathbf{x}, t) = A(\mathbf{x}, t) + 9\, k \log \frac{L}{r} \tag{2}$$

où k est la constante universelle et où $A(\mathbf{x}, t)$ dépend de la macrostructure de l'écoulement. Le facteur 9 est commode pour des calculs ultérieurs.

Mon exposé de la conception de A. M. Obukhov se rangera du côté de [1] et s'appuiera sur la modification correspondante des deux hypothèses de similitude tirées de [1]. Comme troisième hypothèse sera utilisée l'hypothèse déjà formulée de la normalité de la répartition du logarithme ε_r avec la formule (2) pour la dispersion de ce logarithme.

La formule connue

$$B_{dd}(r) = cr^{2/3}\, \bar{\varepsilon}^{2/3}$$

d'après [1] se remplace avec les nouvelles hypothèses par la formule :

$$B_{dd}(r) = C(\mathbf{x}, t) \cdot r^{2/3} \cdot \bar{\varepsilon}^{2/3} \left(\frac{L}{r}\right)^{-k} \tag{3}$$

où k est la constante de (2) et le facteur $C(\mathbf{x}, t)$ dépend de la macrostructure de l'écoulement. Au lieu de la proposition de la constance de l'asymétrie de :

$$S(r) = \frac{B_{ddd}(r)}{B_{dd}^{3/2}(r)}$$

avec $l \ll r \ll L$ formulée dans [2] nous obtenons maintenant

$$S(r) = S_0 \left(\frac{L}{r}\right)^{3/2\,k} \tag{4}$$

où le coefficient S_0 dépend aussi de la macrostructure de l'écoulement.

Lions avec l'échelle de longueur r et avec le point (x, t) les échelles de temps et de vitesse

$$T_r = r^{2/3} \cdot \varepsilon_r^{-1/3}$$
$$U_r = r^{1/3} \cdot \varepsilon_r^{1/3}$$

et l'échelle interne de longueur

$$l_r = \nu^{3/4} \cdot \varepsilon_r^{-1/4}$$

Il est évident que le nombre de Reynolds composé à partir de U_r et r s'exprime par le rapport de $\frac{r}{l_r}$ d'après la formule :

$$R_e = \frac{U_r \cdot r}{\nu} = \left(\frac{r}{l_r}\right)^{4/3} \tag{5}$$

Les coordonnées x'_α, t' du point $(\mathbf{x}', t')$ de la région du point (x, t) s'expriment par les expressions sans dimensions ξ_α et τ

$$x'_\alpha = x_\alpha + \xi_\alpha \cdot r, \qquad t' = t + \tau \cdot T_r$$

Introduisons des vitesses relatives sans dimension

$$V_\alpha(\xi, \tau) = \frac{U_\alpha(\mathbf{x} + \xi r, t + \tau T_r) - U_\alpha(\mathbf{x}, t)}{U_r}$$

Première hypothèse de similitude

Si nous avons $r \ll L$, dans ce cas pour des valeurs données

$$\xi_\alpha^{(k)}, \quad \tau^{(k)}, \qquad \alpha = 1, 2, 3 \qquad k = 1, 2, ..., n,$$

la distribution de probabilité des $3\,n$ grandeurs

$$V_\alpha(\xi^{(k)}, \tau^{(k)})$$

lorsque $R_{e_r} = R_e$

dépend de R_e seulement et est la même dans tous les écoulements turbulents.

Deuxième hypothèse de similitude

Lorsque $R_e \gg 1$

la distribution indiquée dans la première hypothèse ne dépend pas de R_e.

Nous allons désigner les symboles mathématiques absolus (moyennes) par un trait — au-dessus. Comme $\bar{\varepsilon}$ est presque constante dans des régions de petites dimensions en comparaison avec l'échelle extérieure L, dans ce cas, lorsque $r \ll L$, on peut considérer que

$$\bar{\varepsilon}_r = \bar{\varepsilon} \tag{6}$$

Examinons la différence entre les composantes de vitesses extrêmes en deux points se trouvant à une distance r entre eux :

$$\Delta_{dd}(r) = u_1(x_1 + r, x_2, x_3, t) - u_1(x_1, x_2, x_3, t)$$

on voit que

$$\Delta_{dd}(r) = V_\alpha(1, 0, 0, 0) \cdot r^{1/3} \cdot \varepsilon_r^{1/3} \tag{7}$$

Si $r \gg l$

où l est la limite supérieure (à l'exception des cas de probabilité négligeable) de l'échelle interne l_r, dans le cas d'après (6), (7) et les hypothèses I et II il découle que :

$$\overline{|\Delta_{dd}(r)|^3} = C \cdot r\,\bar{\varepsilon} \tag{8}$$

où C est une constante absolue.

Maintenant il faut profiter de la formule montrée par A. M. Obukhov applicable pour les cas où les moments des quantités ont une loi logarithmique de répartition normale.

Voici cette formule :

$$\overline{\zeta^p} = \exp\left(pm + \frac{p^2\sigma^2}{2}\right) \tag{9}$$

où m et σ^2 sont la moyenne et la dispersion de log ζ.

D'après (2), (7), (8) et (9) découle

$$\overline{|\Delta_{dd}(r)|^p} = C_p(\mathbf{x}, t)\,(r\,\bar{\varepsilon})^{p/3}\left(\frac{L}{r}\right)^{k\,\frac{p(p-3)}{2}} \tag{10}$$

en particulier avec $p = 2$

$$\overline{\Delta^2_{dd}(r)} = C_2(\mathbf{x}, t)\,(r\,\bar{\varepsilon})^{2/3}\left(\frac{L}{r}\right)^{-k}$$

c'est-à-dire la formule (3).

De (3) découle (4) parce que la formule dans [2] $B_{ddd}(r) = -\frac{4}{5}\varepsilon r$ reste valable.

Il est possible d'éliminer dans notre exposé le choix particulier de la grandeur $\varepsilon_r(\mathbf{X}, t)$, se trouvant à la base de considérations de A. M. Obukhov. Dans ce cas deux hypothèses se formulent ainsi :

Première hypothèse de similitude

Si $|x^{(k)} - x| \ll L, k = 0,1, ..., n$, la distribution de probabilité des 3 n grandeurs,

$$\frac{U_\alpha(\mathbf{x}^{(k)}) - U_\alpha(\mathbf{x})}{U_\alpha(\mathbf{x}^{(0)}) - U_\alpha(\mathbf{x})} \qquad \alpha = 1, 2, 3 \qquad k = 1, 2, ..., n \tag{11}$$

dépend du nombre de Reynolds seulement

$$R_e = \frac{|U(\mathbf{x}^{(0)}) - U(\mathbf{x})||\mathbf{x}^{(0)} - \mathbf{x}|}{\nu}$$

Deuxième hypothèse de similitude

Lorsque $Re \gg 1$
la distribution indiquée dans la première hypothèse ne dépend pas de Re.

Le contenu des hypothèses supplémentaires de A. M. Obukhov peut être formulé ainsi :

Troisième hypothèse

Deux groupes de grandeurs (II) sont stochastiquement indépendantes, si, dans le premier groupe

$$|\mathbf{x}^{(k)} - \mathbf{x} \geqslant r_1,$$

dans le deuxième groupe

$$|\mathbf{x}^{(k)} - \mathbf{x}| \leqslant r_r \quad \text{et} \quad r_1 \gg r_r$$

Si on veut déduire à partir de la troisième hypothèse la normalité logarithmique rigoureuse de la répartition des différences de vitesses et la formule pour la dispersion des logarithmes de ces différences, identique à celle de (1) et (2), il faut formuler cette hypothèse avec la précision nécessaire.

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Second similarity hypothesis

When $Re \gg 1$, the distribution given in the first hypothesis does not depend on Re.

The essential content of Oboukhov's additional assumptions can be formulated as the

Third hypothesis

Two subsets of values in the set (11) are stochastically independent, if in the first set $|\mathbf{x}^{(k)} - \mathbf{x}| \geqslant r_1$, in the second $|\mathbf{x}^{(k)} - \mathbf{x}| \leqslant r_2$, and $r_1 \gg r_2$.

Naturally the formulation of this hypothesis must be refined if mathematical rigour is desired and if it is to be used to derive the logarithmic normality of the distribution of velocity differences and the formula for the dispersion of the logarithms of these differences analogous to equations (1) and (3).

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A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number*

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The hypotheses concerning the local structure of turbulence at high Reynolds number, developed in the years 1939–41 by myself and Oboukhov (Kolmogorov 1941 *a, b, c*; Oboukhov 1941 *a, b*) were based physically on Richardson's idea of the existence in the turbulent flow of vortices on all possible scales $l < r < L$ between the 'external scale' L and the 'internal scale' l and of a certain uniform mechanism of energy transfer from the coarser-scaled vortices to the finer.

These hypotheses were arrived at independently by a number of other authors and have achieved very wide acceptance. But quite soon after they originated, Landau noticed that they did not take into account a circumstance which arises directly from the assumption of the essentially accidental and random character of the mechanism of transfer of energy from the coarser vortices to the finer: with increase of the ratio $L:l$, the variation of the dissipation of energy

$$\epsilon = \tfrac{1}{2}\nu \sum_{\alpha}\sum_{\beta}\left(\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right)^2$$

should increase without limit. More accurately, it is natural to suppose that when $L/l \gg 1$ the dispersion of the logarithm of ϵ has the asymptotic behaviour

$$\sigma^2_{\log \epsilon} \sim A + k' \log L/l, \tag{1}$$

where k' is some universal constant.

However, Oboukhov has now discovered how to refine our previous results in a way which takes Landau's comments into consideration. The method consists in examining the dissipation

$$\epsilon_r(\mathbf{x}, t) = \frac{3}{4\pi r^3}\int_{|\mathbf{h}|\leqslant r} \epsilon(\mathbf{x}+\mathbf{h}, t)\, d\mathbf{h}$$

* *Editor's footnote.* This paper is a translation (for which the editors take responsibility) of the Russian text of a lecture given by the author at the Colloque International du C.N.R.S. de Mécanique de la Turbulence, held at Marseilles from 28 August to 2 September 1961. The work described in it is related closely to a lecture given by A. M. Oboukhov at another symposium in the following week (see the preceding paper in this journal). The two lectures are being included in the published records of the proceedings of the respective symposia, and are published here also in view of their interest to a large number of English-speaking readers.

averaged for a sphere of radius r, and in assuming that for large L/r the logarithm of $\epsilon_r(\mathbf{x}, t)$ has a normal distribution. It is natural to suppose that the variance of $\log \epsilon_r(\mathbf{x}, t)$ is given by

$$\sigma_r^2(\mathbf{x}, t) = A(\mathbf{x}, t) + 9k \log L/r, \tag{2}$$

where k is a universal constant, the factor 9 is inserted for later convenience, and $A(\mathbf{x}, t)$ depends on the macrostructure of the flow.

I have formulated appropriate modifications to the two similarity hypotheses that I put forward in 1941, and I therefore wish to give an interpretation of Oboukhov's new idea in the light of this earlier work. The assumed normality of the distribution of $\log \epsilon_r$, together with formula (2) for its variance, constitutes a third hypothesis.

I shall show that the equation

$$B_{dd}(r) = C r^{\frac{2}{3}} \bar{\epsilon}^{\frac{2}{3}}$$

from Kolmogorov (1941 *a*) now takes the modified form

$$B_{dd}(r) = C(\mathbf{x}, t)\, r^{\frac{2}{3}} \bar{\epsilon}^{\frac{2}{3}} (L/r)^{-k}, \tag{3}$$

where k is the constant in equation (2) and the factor $C(\mathbf{x}, t)$ depends on the macrostructure of the flow; and that the theorem of constancy of skewness

$$S(r) = B_{ddd}(r)/B_{dd}^{\frac{3}{2}}(r)$$

when $l \ll r \ll L$, derived in Kolmogorov (1941 *b*), is now replaced by the equation

$$S(r) = S_0 (L/r)^{\frac{3}{2}k} \tag{4}$$

where the coefficient S_0 also depends on the macrostructure of the flow.

Let us associate with the length scale r and the point $(\mathbf{x}, t)$ scales of time and velocity

$$T_r = r^{\frac{2}{3}} \epsilon_r^{-\frac{1}{3}}, \quad U_r = r^{\frac{1}{3}} \epsilon_r^{\frac{1}{3}},$$

and an internal length scale,

$$l_r = \nu^{\frac{3}{4}} \epsilon_r^{-\frac{1}{4}}.$$

Obviously the Reynolds number formed from U_r and r is determined by the ratio $r : l_r$ through the equation

$$Re_r = \frac{U_r r}{\nu} = \left(\frac{r}{l_r}\right)^{\frac{4}{3}}. \tag{5}$$

Let x'_α, t' be the co-ordinates of any point $(\mathbf{x}', t)$ in the neighbourhood of of $(\mathbf{x}, t)$. Then we define dimensionless co-ordinates ξ_α and time τ by equations

$$x'_\alpha = x_\alpha + \xi_\alpha r, \quad t' = t + \tau T_r,$$

and dimensionless relative velocities $v_\alpha(\boldsymbol{\xi}, \tau)$ by equations

$$v_\alpha(\boldsymbol{\xi}, \tau) = \frac{u_\alpha(\mathbf{x} + \boldsymbol{\xi} r, t + \tau T_r) - u_\alpha(\mathbf{x}, t)}{U_r}$$

The first similarity hypothesis

If $r \ll L$, then for assigned numbers $\xi_\alpha^{(k)}, \tau^{(k)}$ ($\alpha = 1, 2, 3$; $k = 1, 2, \ldots, n$) the probability distribution of the $3n$ variables $v_\alpha(\boldsymbol{\xi}^{(k)}, \tau^{(k)})$, for the fixed value $Re_r = Re$, depends only on Re and is the same in all turbulent flows.

The second similarity hypothesis

When $Re \gg 1$, the distribution given in the first hypothesis does not depend on Re.

We shall indicate an absolute mathematical expectation (mean) by an overbar. Since $\bar{\epsilon}$ is almost constant in regions which are small in comparison with the external scale L, when $r \ll L$ it may be supposed that

$$\bar{\epsilon}_r = \bar{\epsilon}. \tag{6}$$

Let us examine the difference between the linear components of velocities at two points distant r apart,

$$\Delta_{dd}(r) = u_1(x_1+r, x_2, x_3, t) - u_1(x_1, x_2, x_3, t).$$

From the definitions of v_α and U_r,

$$\Delta_{dd}(r) = v_1(1,0,0,0)\, r^{\frac{1}{3}} \epsilon_r^{\frac{1}{3}}. \tag{7}$$

If $r \gg l$, where l is the upper limit (excluding cases of negligibly small probability) of the internal scale l_r, then from equations (6) and (7) and the above two hypotheses, it follows that

$$\overline{|\Delta_{dd}(r)|^3} = Cr\bar{\epsilon} \tag{8}$$

where C is an absolute constant.

We now use the formula pointed out by Oboukhov for moments of quantities having a logarithmically normal law of distribution, viz.

$$\overline{\zeta^p} = \exp\left(pm + \tfrac{1}{2}p^2\sigma^2\right), \tag{9}$$

where m and σ^2 are the mean and variance of $\log \zeta$. From equations (2), (8) and (9), it follows that

$$\overline{|\Delta_{dd}(r)|^p} = C_p(\mathbf{x}, t)\,(r\bar{\epsilon})^{p/3}\,(L/r)^{\frac{1}{2}kp(p-3)}. \tag{10}$$

In particular, when $p = 2$,

$$\overline{\Delta^2_{dd}(r)} = C_2(\mathbf{x}, t)\,(r\bar{\epsilon})^{\frac{2}{3}}\,(L/r)^{-k},$$

and this is the same as equation (3).

Since the formula

$$B_{ddd}(r) = -\tfrac{4}{5}\bar{\epsilon}r$$

given in Kolmogorov (1941 *b*) remains valid, equation (4) now follows from equation (3).

Our presentation can be freed from the special selection of the values of $\epsilon_r(\mathbf{x}, t)$ forming the basis of Oboukhov's examination. The two similarity hypotheses are then formulated as follows:

First similarity hypothesis

If $|\mathbf{x}^{(k)} - \mathbf{x}| \ll L$, $(k = 0, 1, 2, \ldots, n)$ then the probability distribution of the values of

$$\frac{u_\alpha(\mathbf{x}^{(k)}) - u_\alpha(\mathbf{x})}{u_\alpha(\mathbf{x}^{(0)}) - u_\alpha(\mathbf{x})} \quad (\alpha = 1, 2, 3;\ k = 1, 2, \ldots, n) \tag{11}$$

depends only on the Reynolds number

$$Re = \frac{|u_\alpha(\mathbf{x}^{(0)}) - u_\alpha(\mathbf{x})|\,|\mathbf{x}^{(0)} - \mathbf{x}|}{\nu}$$

Second similarity hypothesis

When $Re \gg 1$, the distribution given in the first hypothesis does not depend on Re.

The essential content of Oboukhov's additional assumptions can be formulated as the

Third hypothesis

Two subsets of values in the set (11) are stochastically independent, if in the first set $|\mathbf{x}^{(k)} - \mathbf{x}| \geq r_1$, in the second $|\mathbf{x}^{(k)} - \mathbf{x}| \leq r_2$, and $r_1 \gg r_2$.

Naturally the formulation of this hypothesis must be refined if mathematical rigour is desired and if it is to be used to derive the logarithmic normality of the distribution of velocity differences and the formula for the dispersion of the logarithms of these differences analogous to equations (1) and (3).

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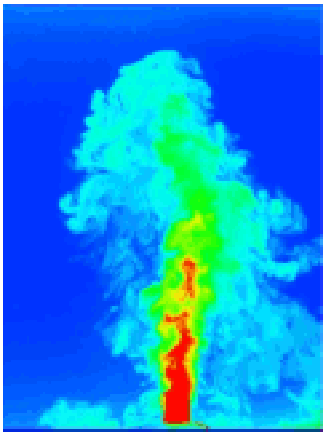

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Homogeneous turbulence: an introductory review

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Homogeneous turbulence: an introductory review

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A brief review of developments in the theory of homogeneous turbulence over the last 50 years is given, many of these developments stemming from lectures and discussions at the 1961 Marseille Colloquium *Mécanique de la Turbulence*. The following topics are discussed: Kolmogorov's 1961 lecture, intermittency and the finite-time singularity problem, the skewness factor paradox, Kraichnan's direct interaction approximation, helicity and the dynamo problem, two-dimensional turbulence, rapid distortion theory and the passive and active scalar problems.

Keywords: Kolmogorov; intermittency; skewness; helicity; rapid distortion

1. Marseille 1961

My credentials for presenting this opening review[1] are slender, relying as they do on the simple fact that I am one of the handful of surviving participants of the now legendary 1961 Marseille Colloque International du CNRS, Mécanique de la Turbulence (hereafter Marseille '61), which this present Colloquium seeks, if not to emulate, at least to commemorate. Allow me first to recall some features of that 1961 Colloquium that made it so special. I draw on the Proceedings [1] as a jog to my memory: a large red volume, perhaps less well-known hitherto than it should have been, but now scanned and accessible on the website <http://turbulence.ens.fr> of this present meeting.

The Colloquium was organised, and the Proceedings edited, by the late Alexandre Favre, Membre de l'Académie des Sciences, who deserves great credit for the memorable character of the event! The Scientific Committee consisted of G.K. Batchelor (my mentor and research supervisor at the time), J. Kampé de Fériet (the 'éminence grise' of French mechanics), L. Kovasznay (USA), H. Schlichting (Germany) and L. Sedov (USSR). In the event, Schlichting and Sedov were not present at the Colloquium, and Kampé de Fériet played a figurehead role, so the burden of scientific organisation fell on Batchelor, Kovasznay and Favre; they invited S. Corrsin, R.W. Stewart and H. Liepmann to act as Chairmen of various sessions – all eminent figures in the subsequent history of turbulence. But particularly memorable was the fact that three great father-figures were also present, namely Theodore von Kármán, G.I. (Sir Geoffrey) Taylor and A.N. Kolmogorov (who was accompanied by three colleagues from the Soviet Union, M.D. Millionschikov, A.M. Obukhov and A.M. Yaglom). This was a unique encounter, during that cold-war epoch, of great scientists from East and West, something that Batchelor had been particularly keen to engineer in

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difficult political circumstances. [The cold war was at its height: the Berlin blockade was in force from July to November that year, when the infamous Berlin Wall was constructed; the Soviet Union resumed nuclear testing on 1st September 1961, during the very week of Marseille'61, in response to a series of four atmospheric nuclear tests carried out by France in the Algerian Sahara between January 1960 and May 1961; and France itself was still, under General de Gaulle, in a state of political turmoil over the Algerian War of independence.]

It is noteworthy that neither Taylor nor von Kármán gave talks at the Colloquium, but they of course contributed to the discussions, and their presence lent great authority to the deliberations. Neither did Favre lecture, but his coordinating role as host was crucial. As for Millionschikov, he later rose to high office in the Soviet bureaucracy (a USSR postage stamp bearing his image was issued in 1974!), and his presence was sufficient to ensure that no member of the KGB was needed to 'keep an eye' on the Soviet delegation.

The second session of Marseille '61 was devoted to *Energy Transfer in Homogeneous Turbulence*, and it is on aspects of homogeneous turbulence that I focus in this review. Homogeneous turbulence, i.e. turbulence whose statistical properties are invariant under translations of the frame of reference, is of course a fiction: no field of turbulence is strictly homogeneous, since the presence of fluid boundaries inevitably introduces a degree of inhomogeneity. Nevertheless, 'homogeneity' is a useful idealisation that can be approximately realised in wind tunnel 'grid-generated' turbulence, widely studied in the belief that this provides a legitimate description of at least the smaller scales of turbulence far from fluid boundaries. For turbulence that is strictly homogeneous, the abstract concept of the 'ensemble average' may be conveniently replaced by the more accessible concept of the 'space average'; ergodic theory tells us that these two averages are, under the condition of homogeneity, equivalent.

Turbulence that is homogeneous may also be 'isotropic', i.e. its statistical properties may be invariant under rotations, as well as translations, of the frame of reference. In such a field of turbulence, there is no 'preferred direction': all directions are statistically equivalent. It should be noted however that isotropic turbulence need not be invariant also under mirror reflexions, a distinction that becomes crucially important when magnetohydrodynamic effects are under consideration (see §6 below).

Homogeneous turbulence may be forced or unforced. Forcing can be achieved in numerical simulations, though not in laboratory experiments, by inclusion of a random body force $\mathbf{f}(\mathbf{x}, \mathbf{t})$, itself statistically homogeneous, in the Navier–Stokes equation governing the flow. If this body force is also statistically stationary in time, a statistically steady homogeneous turbulent state can result; in this case, the ensemble average is equivalent to a time average, which is both conceptually clear and computationally convenient.

Unforced homogeneous turbulence, on the other hand, necessarily decays in time, as a result of the non-linear cascade of energy to the very small scales of motion at which viscous dissipation of kinetic energy sets in. It is an essential tenet of Kolmogorov's [2] theory that the mean rate of cascade (and subsequent decay) of energy ϵ is independent of kinematic viscosity ν in the limit $\nu \to 0$.

2. K41 and Kolmogorov's 1961 lecture

This 1941 theory of Kolmogorov (K41) was then, as still even now, a central pillar of understanding of the mechanics of turbulence, and so it was particularly fitting that he should be present at Marseille '61. In fact, a highlight of the meeting, alluded to only in a footnote to the paper by T.H. Ellison (p. 115 of the Proceedings), was the display of slides by Stewart from the observations later published in *JFM* [3] of turbulent spectra in a tidal channel at Reynolds number (based on depth and mean velocity) up to 3×10^8. These experiments provided what Ellison described as 'by far the most convincing demonstration of the correctness of the Kolmogorov theory that has yet been made'. This was of exceptional importance for two reasons: first, because huge effort had been expended, particularly by Batchelor (and references therein) [4] in promoting the Kolmogorov theory, and much of the current understanding relied on the Kolmogorovian concept of the 'energy cascade' and the $k^{-5/3}$ energy spectrum to which this gave rise; and second, because this picture was being currently challenged by the alternative 'direct-interaction' theory of Kraichnan (1959) [5] which led to a significantly different ($k^{-3/2}$) spectrum. The tidal-channel observations reported by Stewart gave unambiguous support for the Kolmogorov scenario.

So far, so good! But then Kolmogorov was invited to give his lecture with the title 'Précisions sur la structure locale de la turbulence dans un fluide visqueux aux nombres de Reynolds élevés'. The paper is reproduced in both French and Russian in the Proceedings, and in English translation in *JFM* [6]. Kolmogorov delivered his lecture in French, and his only visual aid was the blackboard. What was so startling was that (following reference to parallel work of Obukhov and to comments that he attributed to Landau) Kolmogorov proceeded to point out fundamental inconsistencies in his own (1941) theory, in that this theory failed to take account of the spatial fluctuations in the rate of turbulent dissipation of energy ϵ, the key parameter of the theory. Kolmogorov offered arguments supporting a log-normal probability distribution for this quantity, and showed that the exponent in the energy spectrum would be slightly modified through this 'intermittency' effect. (Later work on the behaviour of higher-order structure functions of the form $S_n(r) = <(u(\mathbf{x}+\mathbf{r}) - u(\mathbf{x}))^n>$, for $n = 2, 3, 4, \ldots$, where u is, say, the velocity component parallel to the direction of the separation vector $\mathbf{r}$, has revealed that Kolmogorov's log-normal assumption itself is untenable ([7,8]).)

This lecture, Kolmogorov's final contribution to turbulence, was, to put it mildly, a bit of a bombshell! There had been hints previously that all was not well with the Kolmogorov picture; in particular, as described in Batchelor's [4] monograph, the dimensionless flatness factor of the velocity derivative

$$F = \frac{\langle(\partial u/\partial x)^4\rangle}{\langle(\partial u/\partial x)^2\rangle^2}, \tag{1}$$

which, on the Kolmogorov theory should be a universal constant in the limit of large Reynolds number Re, in fact showed a worrying experimental increase with increasing Re, with no tendency to settle down to a limiting constant value. The implication was that, as Re increases, the vorticity of the turbulence tends to become more and more concentrated in 'fine structures' (vortex tubes and/or sheets being the most natural candidates), a picture that was hard to reconcile with the idea of an energy cascade from the energy-containing scale l_0 through a geometric sequence of length scales like $2^{-n}l_0$. Kolmogorov's revised theory, taking account of the intermittency of $\epsilon(\mathbf{x}, t)$, was however potentially capable of explaining the increase of F with Re, and even more so, of the behaviour of higher-order statistical properties of the small-scale features of the turbulence (including in particular the vorticity field).

3. Intermittency and the finite-time singularity problem

Kolmogorov's paper marked the onset of a widespread effort in the turbulence research community to understand the phenomenon of intermittency, for which many competing models have been proposed. This has been much stimulated on the one hand by the detection of concentrated vortices within a field of turbulence [9] and on the other by direct numerical simulations (DNS), which display turbulence as a random superposition of concentrations of vortex tubes on all scales from the Taylor microscale $\sim l_0 Re^{-1/2}$ down to the Kolmogorov scale $\sim l_0 Re^{-3/4}$ (e.g. [10,11]). The cascade picture has now given way to a picture in which the progressive degradation of energy from large to small scales is simply a consequence of vortex tube stretching, each vortex tube being locally stretched (with associated decrease of cross-section, i.e. decrease of scale) by the straining flow induced by all the other vortices of the field of turbulence.

This revised picture has in turn re-stimulated interest in the fundamental problems of regularity associated with the Euler equations and the Navier–Stokes equations at very high Re as first addressed by Leray [12]: given smooth initial conditions for the vorticity field, does this field in general remain smooth for all time, or alternatively, does there exist any smooth initial condition of finite energy from which a singularity of vorticity emerges within a finite time? The question is of course fascinating in its own right, and quite independently of the general problem of turbulence. The natural initial condition to consider is a vortex-tube tangle, each tube containing axial vorticity of Gaussian cross-section. Is it not amazing that we still do not know whether such tangles can lead to a singularity or not! DNS is as yet inconclusive, in no small part because numerical techniques inevitably break down as a singularity is approached. If a singularity can indeed form, then some kind of self-similar structure (or structure within a structure) of scale decreasing to zero at the singularity time is to be expected, but such description remains elusive.

The problem is important for turbulence, because it has an obvious bearing on the structure of turbulence at the smallest scales. Considering for the moment the inviscid limit $\nu \to 0$, the Kolmogorov spectrum extends to wavenumber $k = \infty$ (i.e. zero length-scale). The time-scale for energy transfer from scale k^{-1} to scale zero, as determined dimensionally from ϵ and k, is then $t_k = \epsilon^{-1/3} k^{-2/3}$. This suggests that in the Euler limit, singularities must develop at finite time. Of course, a divergent enstrophy spectrum $\sim k^{1/3}$ suggests the same. But no explicit solution of the Euler equations has as yet revealed the singular asymptotic structure of the vorticity field that such arguments might suggest.

DNS has played an increasingly prominent role in turbulence research ever since the pioneering work of Orszag and Patterson [13] (see, for example, the recent review by Ishihara et al. [14]), and its importance cannot be over-estimated. It remains fair to say however that advances in understanding require a symbiotic interaction between DNS and theoretical analysis: each activity feeds on, and at the same time provides guidance for, the other; now only in this way can further real progress be made.

4. The paradox of the skewness factor

A related puzzle concerns the skewness factor of the velocity derivative

$$S = \frac{\langle(\partial u/\partial x)^3\rangle}{\langle(\partial u/\partial x)^2\rangle^{3/2}}, \tag{2}$$

which again, on K41 theory, should be a universal constant in the limit $Re \to \infty$. Batchelor said in his Marseille '61 lecture that 'the available measurements ... suggest that it is still decreasing (in magnitude) at the highest Reynolds numbers of the measurements, and we cannot yet be sure that the asymptotic value is non-zero'. This remains a crucial issue, because in 3D isotropic turbulence the vorticity $\boldsymbol{\omega}$ satisfies

$$\left\langle |\boldsymbol{\omega}|^3 \right\rangle \sim |S| \left\langle \boldsymbol{\omega}^2 \right\rangle^{3/2} , \tag{3}$$

and the enstrophy equation takes the form

$$\frac{\partial \left\langle \boldsymbol{\omega}^2 \right\rangle}{\partial t} = C \left\langle \boldsymbol{\omega}^2 \right\rangle^{3/2} - 2\nu \left\langle (\nabla \times \boldsymbol{\omega})^2 \right\rangle , \tag{4}$$

where C is a dimensionless positive constant proportional to $|S|$. In the formal limit $\nu = 0$, this equation implies blow-up of enstrophy in a finite time, and again, this would appear to imply the appearance of singularities in the vorticity distribution, at least in the Euler limit. So, what is the latest information on the skewness factor in the high Re limit? I do not know that we are any nearer to a definite answer to this central question than we were in 1961.

5. Kraichnan's direct interaction approximation

Bob Kraichnan also gave a memorable lecture at Marseille '61 in which he compared various approaches to the closure problem of turbulence, including the quasi-normal (or zero-fourth-cumulants) approximation of Millionschikov [15] and his own direct-interaction approximation published two years earlier [5]. Kraichnan simply stood at the microphone and spoke for 40 minutes with extraordinary lucidity, and without the help of a single visual aid, even chalk on blackboard! His lecture was followed by a critique of 'Kraichnan's theory of turbulence' by Ian Proudman, a younger colleague of Batchelor, who had published an extensive treatment of the quasi-normal theory with W.H. (Bill) Reid some years earlier ([16]; see also [17])[2]. Proudman's critique was savage by any standard, his conclusion being: 'I do not hold out much hope for useful results from the theory [of Kraichnan]'. Kraichnan was understandably perturbed by this unfavourable assessment.

In some respects, Proudman's pessimism was justified. As mentioned earlier, the experimental evidence did not support a $k^{-3/2}$ energy spectrum, which was a key prediction of the theory. However, in his response (published as an Appendix to his own paper in the Proceedings), Kraichnan pointed out a key 'realisability' property of his theory, namely that it guaranteed that the energy spectrum would remain positive for all time. He referred to the paper by Ogura (presented at the IUGG-IUTAM Symposium that followed immediately on Marseille '61, and later published as Ogura [19]), which provided numerical evidence that the quasi-normal closure (which incidentally leads to an enstrophy equation of the form (4)) leads to the inadmissible development of a negative energy spectrum in the energy-containing range of wave-numbers. In this respect therefore, the direct interaction approximation was clearly superior (and this because it was an exact closure for a physically realisable dynamical system, although unfortunately not the Navier–Stokes system).

Kraichnan was subsequently able to modify his theory in such a way as to yield a $k^{-5/3}$ spectrum (his Lagrangian-History Direct Interaction Approximation, or LHDIA). I spent four months in Stan Corrsin's group at Johns Hopkins University (September–December 1965), and I recall that we met for a weekly sandwich-lunch discussion session, to work our way through this theory, but such was its complexity that we were not much the wiser by the end of the term. The theory [20] takes account of the 'sweeping' of small eddies by large eddies, in a mixed Lagrangian–Eulerian framework. In effect, this forces the energy transfer to be local in wave-number space, and this in turn leads on dimensional grounds to the Kolmogorov spectrum $E(k) = C\epsilon^{2/3}k^{-5/3}$. The reward for all the hard work is that the universal constant C is determined by the theory; the downside is that, although $k^{-5/3}$ appears fairly robust, the 'constant' C varies by about $\pm 30\%$ in different experimental contexts (Monin and Yaglom [21], Chap. 8).

6. Helicity and the dynamo problem

One important aspect of homogeneous turbulence is the manner in which it acts upon a convected scalar field (like temperature) or vector field (like magnetic field), in conjunction with the effects of molecular diffusion. The passive scalar problem is to be covered in another section of this meeting, so I shall say no more here (except to recall that Philip Saffman gave a remarkable lecture at Marseille '61 on the counter-intuitive interaction of molecular and turbulent diffusion of a scalar released from a fixed point in a turbulent flow – Saffman [22]). But I would like to comment on the vector field problem, because many of the advances in our understanding of the dynamics of turbulence have come from parallel considerations concerning MHD turbulence.

The emphasis in 1961 was on the increasing degradation of energy in going from large to small scales, i.e. the production of chaos (the small-scale turbulence) from order (the large-scale shearing or straining motion). A dramatic change came from an unexpected quarter in the paper of Steenbeck et al. [23][3]. These authors recognised that a large-scale well-ordered magnetic field could arise more or less spontaneously as a result of coherent interaction of small-scale random motion and the small-scale random field fluctuations to which this motion gives rise; in other words, order could arise out of chaos. I see this now as the most far-reaching and important discovery in the context of turbulence of this last half-century; but I have to admit to a certain degree of prejudice, because I came into this topic myself quite independently from a different direction three years later. I had been intrigued by the paper of Woltjer [26] who had shown that under certain circumstances, the quantity

$$H_M = \int \mathbf{A} \cdot \mathbf{B}\, dV \tag{5}$$

(where $\mathbf{A}$ is a vector potential for $\mathbf{B}$) is an invariant in a perfectly conducting fluid. The key was to recognise that the reason for this invariance is simply that the lines of force are transported with the fluid, and their topology is therefore conserved. It was a small step to realise that the same type of invariant must exist for the vorticity field in an ideal fluid in which vortex lines are 'frozen' in the flow. This 'inviscid invariant' is the helicity

$$H = \int \mathbf{u} \cdot \boldsymbol{\omega}\, dV, \tag{6}$$

since $\mathbf{u}$ is, in effect, a vector potential for $\boldsymbol{\omega}$. This new invariant [27], [28] is a consequence of Kelvin's [29] theorem on the conservation of circulation; but it had lain undiscovered for close on 100 years since then!

It might be thought that helicity conservation, in conjunction with energy conservation, should have a profound effect on the cascade process through the inertial range (where viscous effects are negligible). Indeed, if we think of this energy cascade in terms of vortex tube stretching, then conservation of helicity does enter the picture, implying as it does, that vortex tube topology is conserved. The effect is less obvious in spectral terms; the fact that the helicity spectrum is not positive-definite means that one cannot argue for a simple cascade of helicity from large to small scales: helicity can be created at large scales at the expense of the creation of equal and opposite helicity at small scales!

The presence of non-zero helicity does however have a profound effect on the generation of a large-scale magnetic field (the dynamo effect) as Steenbeck et al. had predicted. Even at low magnetic Reynolds number Re_m, magnetic field will grow under the effect of homogeneous turbulence provided only that the helicity is non-zero (providing an ‘alpha effect’), and the space available for the growth of the field is large enough. This space requirement is easily satisfied in planetary and astrophysical contexts! There is also an enhanced diffusivity due to the turbulence (the ‘beta effect’), but the alpha effect dominates on a large enough scale [30].

In his Marseille ’61 lecture, ‘Turbulence in compressible and electrically conductive media’, Kovasznay anticipated this turbulent diffusivity effect. He also emphasised the need for a ‘clear, simple and meaningful experiment’ in order ‘to obtain educated guesses about relative importance of different approaches’. Such an experiment has been brilliantly conceived and realised by the French team working at Paris (ENS and Saclay), Lyon and Cadarache (Monchaux [31]) have demonstrated dynamo action in a ‘washing machine’ in which liquid sodium is stirred into strongly helical turbulent motion by counter-rotating propellers. Both mean flow and turbulent dynamo effects are present in this experiment, which has for the first time demonstrated that the process responsible for generation of the Earth’s magnetic field can indeed be simulated in the laboratory.

7. Two-dimensional turbulence

Two-dimensional turbulence was briefly discussed by Batchelor in the closing sections of his 1953 monograph, where he pointed out an important consequence of (inviscid) enstrophy conservation: the inevitability of an inverse energy cascade (although he did not use this terminology). I believe his interest in this problem was rekindled through discussions with Kraichnan at Marseille ’61. At any rate, soon after, he put his research student Roger Bray to work on a numerical simulation of 2D turbulence [32]. More or less simultaneously, Kraichnan developed his ideas on the subject. Kraichnan [33] and Batchelor [34] gave his version of the enstrophy cascade, with reference to Bray’s numerical results. The inverse energy cascade can also be regarded as a manifestation of the emergence of ‘order out of chaos’ as later clarified by McWilliams [35] in his paper ‘The emergence of isolated coherent vortices in turbulent flow’. The importance of nearly 2D turbulence in the context of atmospheric dynamics has led to intense activity in this field. Density stratification, strong rotation with associated Coriolis effects, and strong ‘applied’ magnetic field can all (separately or possibly in conjunction) lead to ‘two-dimensionalisation’ of a field of turbulence, although the residual effect of weak three-dimensionality can remain of crucial importance. Such turbulence is strongly anisotropic, but nevertheless can remain homogeneous; it is a subject of continuing research interest in which real progress may be anticipated.

8. Rapid distortion theory

Homogeneous turbulence that is subjected to uniform strain, whether rotational or irrotational, remains homogeneous, although increasingly anisotropic as the effect of the strain progresses. If this straining is sufficiently rapid, then its effect on the turbulence may be treated in at least a first approximation by a linearised analysis in which nonlinear interaction of the turbulence with itself is neglected. This is what is known as rapid distortion theory (or RDT). Like so many aspects of turbulence, RDT goes back to a paper of G.I.Taylor [36] who analysed the damping of turbulent fluctuations in flow through a contraction in a wind tunnel.

In his 1956 book, *The Structure of Turbulent Shear Flow* [37], Alan Townsend had used a version of RDT to determine the structure of what he described as the 'big eddies' in free turbulent flows. He assumed that only the irrotational part of the strain was relevant and allowed this strain to progress only until a degree of anisotropy consistent with experimental measurements was attained. In his Marseille '61 lecture on 'Free turbulent flows', Hans Liepmann made passing reference to this approach, but regarded the resulting models of eddy structure as but 'a groping for an eventual representation of a stochastic rotational field'.

I could not understand Townsend's reason for focusing on just the irrotational ingredient of the strain field, and in a later investigation I found that the effect of plane shear on turbulence was indeed quite different [38]: as is now well recognised, plane shearing is the mechanism by which streamwise vortices become prominent in the flow. S.J. Kline discovered the related 'streaks' in a turbulent boundary layer at about the same time [39]. 'Coherent structures' became a growth industry during the 1970s, a period that marked a return towards mechanistic approaches based on vortex interactions in **x**-space, in parallel with the purely statistical approach that found its most natural expression in **k**-space.

In the second edition of his book (1976, §3.12 *et seq.*), Townsend adopted the plane-shear (as opposed to the irrotational-strain) description in his approach to the energy-containing eddies, and concluded: 'Generally, the measurements of the behaviour of turbulent fluid under either irrotational distortion or plane shearing flow show that the larger eddies, possessing most but not all of the energy, are persistent and coherent structures which interact with the mean flow in the "rapid-distortion" manner, and which lose energy and interact with the smaller eddies by processes that can be modelled by an effective viscosity'. The fact that linearised RDT is more successful than we have any right to expect provides one ray of light in the continuing search for a complete theoretical description of turbulence.

Additional effects such as uniform density stratification, Coriolis forces due to rotation, and Lorentz forces due to an applied uniform magnetic field can all act in conjunction with uniform distortion to induce anisotropy in a homogeneous field of turbulence, without inducing inhomogeneity. Much work in this area has been described in the book *Homogeneous Turbulence Dynamics* by Sagaut and Cambon [40], which testifies to the continuing vitality of RDT, particularly in geo- and astro-physical contexts.

9. The active scalar field problem

If weak variations of a scalar contaminant are present in a field of turbulence, then this scalar field is subject to convection and molecular diffusion, but does not react back dynamically upon the convecting velocity field; this constitutes the 'passive scalar field problem'.

There are many circumstances however in which the contaminant *may* react back upon the velocity field. For example, if the 'contaminant' is temperature $\theta(\mathbf{x}, t)$, then buoyancy forces come into play, and the velocity field $\mathbf{u}(\mathbf{x}, t)$ is itself influenced to a greater or lesser extent by the θ-field.

In an extreme situation, we may suppose that $\mathbf{u}$ is actually 'driven' by the θ-field. This was the situation considered by Batchelor et al. [41], in their paper *Homogeneous buoyancy-generated turbulence* (Batchelor's last significant contribution to turbulence). The non-linearity $\mathbf{u} \cdot \nabla\theta$ associated with convection of θ is now just as important as the nonlinear contribution $\mathbf{u} \cdot \nabla\mathbf{u}$ to the Navier–Stokes equations.

If we consider turbulence in a rotating system at small Rossby number (i.e. dominant Coriolis effect, as in the Earth's liquid core), then the non-linear acceleration $\mathbf{u} \cdot \nabla \mathbf{u}$ is negligible, and we have a form of turbulence driven by buoyancy forces in which the sole nonlinearity is represented by the convective term $\mathbf{u} \cdot \nabla \theta$ in the advection–diffusion equation, a state of affairs that may be described as 'geostrophic turbulence'. If a linearised Lorentz force is also present, then we have 'magnetostrophic turbulence' which again (starting from a very weak field) is potentially capable of dynamo action [42]. If the statistics of the θ-distribution are assumed known, then the statistics of the velocity and magnetic fields can be deduced by linearised analysis similar to RDT. The essential problem that remains is to determine the statistical evolution of the θ-field, given the source of temperature variations, a problem whose regularity properties have recently been investigated by Friedlander [43]. This type of turbulence problem is far removed from conventional laboratory turbulence; but it contains the same essential ingredients: non-linear spreading of energy in **k**-space, and dissipation by molecular diffusivity at the smallest scales. Such problems continue to present a challenge for the future.

10. Conclusion

The Marseille 1961 Turbulence Colloquium was remarkably wide-ranging in the issues that it explored, and the discussions at the Colloquium sowed the seeds for a number of important later research developments. Computers were still at a relatively primitive stage of development, but even then the numerical work of Ogura [19] that demonstrated a fundamental flaw of the quasi-normal approximation gave a hint of the central role that computation was to play in the future. The search for a fundamental theory of turbulence was very much alive then; it is still alive today, though perhaps in a slightly battered state, since so many approaches have been tried and discarded along the way. The fact that order can emerge from chaos in MHD turbulence (through the spontaneous growth of a large-scale magnetic field) is, in my view, one of the most striking discoveries of the last 50 years, as is the identification of concentrated vortices as perhaps the key element of turbulent flow. The interaction of these vortices remains a pressing problem, both for theory and numerics, and the underlying problem of regularity of the Navier–Stokes equations (otherwise known as the finite-time singularity problem) remains a central challenge for mathematically oriented fluid dynamicists, in essential partnership with experts in scientific computing. Let us hope the next 50 years will reveal the answer to this problem, one way or another!

Notes

1. Lecture delivered at the Turbulence Colloquium, Marseille 2011, commemorating Mécanique de la Turbulence, Marseille 1961.
2. A new 'cross-independence' closure theory has recently been proposed by Tatsumi [18], 54 years after the same author's first paper on the decay of isotropic turbulence, a remarkable span of sustained investigation!
3. This paper and subsequent papers by the same authors were published in German, and were not very well known in the West until translated into English by Roberts and Stix [24], and subsequent publication of the book of Krause and Rädler [25].

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Turbulence: the chief outstanding difficulty of our subject

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Abstract. A review of interesting current topics in turbulence research is decorated with examples of popular fallacies about the behaviour of turbulence. Topics include the status of the Law of the Wall, especially in compressible flow; analogies between the effects of Reynolds number, pressure gradient, unsteadiness and roughness change; the status of Kolmogorov's universal equilibrium theory and local isotropy of the small eddies; turbulence modelling, with reference to universality, pressure-strain modelling and the dissipation equation; and chaos. Fallacies include the mixing-length concept; the effect of pressure gradient on Reynolds shear stress; the separability of time and space derivatives; models of the dissipation equation; and chaos.

1 Introduction

Those who knew the British applied mathematician Keith Stewartson (1925–1983) will recall that his strongest term of scientific condemnation was "unrigorous". I'm sure he regarded the whole phenomenon of turbulence as being unrigorous and probably invented by the Devil on the seventh day of Creation (when the Good Lord wasn't looking); I am inclined to agree. Keith would certainly have approved of the rigour of Horace Lamb's Hydrodynamics – what the reviewer of a later book called his "awful correctness". However Lamb, after discussing all the branches of hydrodynamics known to him, finally had to deal with turbulence and remarked, in Article 365, p. 651 of the 4th edition (Lamb 1916), "It remains to call attention to the chief outstanding difficulty of our subject." Seventy-odd years have come and gone; difficulties in hydrodynamics have come and gone; but turbulence still remains as the "chief outstanding difficulty of our subject". The late Jack Nielsen, Chief Scientist of NASA Ames Research Center, said a few years ago that turbulence modeling was the "pacing item" in the use of the NAS computer complex at Ames: his comment, like Lamb's, is still true.

In the last ten years or so we have become able to solve the complete time-dependent Navier-Stokes equations for turbulent flow. However, the Reynolds numbers at which we can get numerically-accurate complete solutions are usually only about three or four times the lowest at which turbulence can exist, and are considerably lower than the Reynolds numbers obtainable in laboratory experiments, let alone those found in real life. Therefore, although turbulence is starting to become accessible to computers, there is no immediate prospect of the subject going the same way as stress analysis and succumbing almost entirely to computation: unlike elasticity, turbulence is a nonlinear (strictly, quasi-linear) phenomenon and, at least at high Reynolds numbers, is at present accessible only to experiment. Thus, experimental fluid dynamics will last for many years more.

Of course, turbulence would merely be a laboratory curiosity or a computational playground if it were not for its extreme importance in real life and in most scientific and engineering disciplines:- in meteorology, aeronautical aerodynamics, shipbuilding, oceanography, in all forms of pipeline design and manufacture, in combustion, in any form of mixing of contaminant, whether of heat or concentration or pollutant – in other words in almost all forms of "interesting" fluid motion except those on an extremely small scale. The cream poured into a cup of coffee goes turbulent, and the flow patterns look very cloud-like.

This paper is intended to inject a certain amount of rigour into the study of turbulence, specifically by reviewing some popular fallacies about turbulence and the way in which turbulent flows behave. Some of these fallacies or illogicalities are propagated by popular but outdated textbooks, but some are at a deeper level of incomprehension, including the preconceptions of workers in statistical mechanics who think that turbulence must be easy. Parts

This paper was originally presented, as the Third Stewartson Memorial Lecture, at the Symposium on Numerical and Physical Aspects of Aerodynamic Flows, Long Beach, 1992. The present version includes some new material and omits some jokes.

of the material are controversial, in the sense that some of my professional colleagues may disagree with me. However, even the controversial sections of the paper may be of interest and stimulate clarifying discussion. It is of course difficult to group illogicalities into any logical order, so I have imbedded them into a study of the more popular topics of turbulence "theory".

My favorite definition of turbulence is that it is the general solution of the Navier-Stokes equations. This is the perfect answer by a government servant to an inquiry by a Congressman or Member of Parliament: it is brief, it is entirely true, and it adds nothing to what was known already. Nearly everybody believes, of course, that the Navier-Stokes equations are an adequately exact description of turbulence, or indeed of any other nonrelativistic motion of a Newtonian fluid. Even the smallest eddies in turbulence in ordinary liquids and gases at earthbound temperatures and pressures are large compared to the mean free path between molecular collisions, so the constitutive equation of the fluid is not in doubt. However, Sect. 4 of the present paper deals with the influence of fluctuating dilatation div u on turbulence in compressible gas flow, and in this case the uncertain value of the bulk viscosity β (Goldstein 1972) may matter.

When people regret that we do not "understand" turbulence they are really regretting that we are not able to integrate the Navier-Stokes equations in our heads: for that, one would need a Cray in the cranium.

It is generally accepted that the basic phenomena of turbulence are the same at any Mach number – except for some special effects to be discussed in Sect. 4 – so unless stated otherwise density will be assumed to be constant.

2 The Law of the Wall

One of the main building blocks, or even foundation stones, of the engineering study of turbulence is the "Law of the Wall". It derives from the hypothesis/assumption that, sufficiently close to a solid wall (meaning, for example, a distance from the wall an order of magnitude less than the diameter of a pipe or the thickness of a boundary layer) the flow depends only on the distance from the wall, on the shear stress at the wall τ_w, and on fluid properties. The characteristics of the outer part of the flow do not matter except that they determine τ_w. (In the discussion below, the term "shear stress" will sometimes be used to mean "shear stress $\div$ density", for short.)

Let us consider a boundary layer for simplicity. The characteristics of the outer part of the flow include the free-stream velocity U_e and the boundary layer thickness δ. The irrelevance of U_e, as such, is a consequence of Galilean (translational) invariance and does not need much discussion. The irrelevance of δ is more crucial, as it depends on the assumption that the flow close to the surface consists of eddies whose length scales (in all directions) are proportional to y, with negligible contributions from eddies whose length scales (in any direction) depend on δ; if this is so, the boundary layer thickness should not appear in any scaling of the inner-layer eddies. We shall see in Sect. 5 that this hopeful view is not quite correct, but it is certainly acceptable to first order.

The consequence of these arguments is, of course, that the mean velocity and turbulence near the surface should scale on the "friction velocity" $u_\tau \equiv (\tau_w/\rho)^{1/2}$, on the distance from the surface, y, and on the kinematic viscosity ν. One of the several dimensionally-correct ways of writing this relationship is

$$U/u_\tau = f_1(u_\tau y/\nu). \tag{1}$$

Another is obtained by differentiating Eq. (1) and hiding a factor of $u_\tau y/\nu$ inside the function f_2, as

$$\partial U/\partial y = (u_\tau/y) f_2(u_\tau y/\nu). \tag{2}$$

Here $u_\tau y/\nu$ is an eddy Reynolds number based on the eddy velocity and length scales, i.e. the friction velocity and the distance from the surface. At large values of this Reynolds number (in practice, about 30–50) we expect the effects of viscosity on the turbulence to be negligible so that Eq. (2) reduces to

$$\partial U/\partial y = u_\tau/(\kappa y) \tag{3}$$

where $\kappa \approx 0.41$ is a constant – Von Karman's constant. The integral of this relationship is the logarithmic law, the additive constant $C \approx 5$ being a constant of integration depending on the velocity difference between the wall and the point at which Eq. (3) becomes valid.

The advantages of the above analysis over the traditional "overlap" demonstration are (i) that the only assumption made about the outer layer is that it doesn't matter, and (ii) that a simple physical argument can be used to simplify Eq. (2) to the so-called mixing-length formula, Eq. (3).

The constant of integration C is equal to 5 only on smooth walls: on rough walls, it becomes a function of the roughness Reynolds number $u_\tau k/\nu$ and of the roughness geometry; the uncertainty of the effective origin of y on rough walls is a further complication. The constant κ, on the other hand, is supposedly universal: it is the same in flows of water and of air on all geometries involving smooth surfaces, and indeed on all geometries involving only small roughness; it is the same in the atmospheric boundary layer, in the depths of the ocean and on the sands of Mars. Alas κ and C are not constant within the turbulence modelling community – a remarkably wide range of values is in use. Those quoted are from the painstaking data analysis of Coles (1969).

Now there are still textbooks – and even living people – that regard the log law as a deduction from the mixing-length formula, Eq. (3), (which it is) and also regard the mixing-length formula for the inner layer as correct (which it is) and also regard Prandtl's original derivation of the mixing-length formula by analogy with molecular motion as correct (which it certainly is not). As the Roman Catholic Church, quoting Aristotle, very properly pointed out to Galileo, the success of deductions from a hypothesis does not prove its truth. Philosophers call this the fallacy post hoc, ergo propter hoc ("after that, therefore because of that") and it is the basis of witch-doctoring (last time we slaughtered a white cow, it rained; there is a drought; therefore . . .).

Quite apart from philosophical questions of falsifiability, it is clear that if a result can be derived by dimensional analysis alone, like Eq. (3), then it can be derived by almost any theory, right or wrong, which is dimensionally correct and uses the right variables. There is a strong suspicion (Bradshaw 1974) that Prandtl got the idea of the lumps of fluid ("Flüssigkeitsballen") of mixing-length theory from visual studies of turbulent open-channel flows with particles sprinkled on the surface to show up the motion. Unfortunately the boundary condition at a free surface permits only motion tangential to the surface and not normal to it, so the surface becomes a plane of symmetry with the vorticity vector everywhere normal to it. The only motions that can remain are the two-dimensional whirls in the surface which sailors, but not landlubberly turbulence researchers, call "eddies". Try it, and you will see what Prandtl saw.

3 Extensions to the law of the wall

The law of the wall derived in Sect. 2 is supposed to be valid for a shear stress equal to the wall shear stress and a density equal to the wall density. In boundary layers in small pressure gradient, the shear stress gradient near the wall is even smaller (the difference being the acceleration) and the layer of approximately constant stress extends out at least as far as the position where the flow ceases to scale on y alone. There is some support for an extended version of Eq. (3), still for $u_\tau y/\nu > 30$ approx., in conditions where either the shear stress $\tau \equiv -\overline{\rho uv}$ or the density ρ varies with distance from the surface. If u_τ is replaced by $(\tau/\rho)^{1/2}$, we get

$$\partial U/\partial y = (\tau/\rho)^{1/2}/(\kappa y) \tag{4}$$

The hand-waving argument for Eq. (4) is that, in the original analysis leading to Eq. (3), u_τ is really being used as the velocity scale at height y, and not as a true surface parameter: if τ varies with y then the local value, rather than the wall value, is the correct one to use in formulating an eddy velocity scale. This would be a rigorous argument only if the typical eddy size were small compared with y, so that the local shear stress would be closely equal to the right basis for a velocity scale, namely some kind of weighted-average shear stress over a y distance equal to a typical eddy size. Unfortunately, of course, the eddy size is of the same order as y.

All we can claim is that local shear stress gives the best easily-available velocity scale. Therefore, the extension of Eq. (3) to Eq. (4) requires an extension of faith in the inner-layer hypothesis which by no means all research workers possess. Nevertheless the application of Eq. (4) to flows with suction or injection, where the shear stress varies with distance from the surface according to $\tau = \tau_w + \rho U V_w$, is fairly well supported by the limited experimental data: almost all the existing correlations are based on the data of Simpson (1967) – and are more scattered than the data! An operational difficulty is that in typical flows with suction or injection the surface is porous on a length scale, h say, which is usually not small compared with the viscous scale ν/u_τ, so that the "roughness" or "porosity" Reynolds number $u_\tau h/\nu$ is important, implying that the additive constant in any integral of Eq. (4) will depend on the surface conditions as well as on the transpiration parameter V_w/u_τ.

Near a sharp trailing edge, the law-of-the-wall arguments can be formally extended to

$$U/u_\tau = f_1(u_\tau x/\nu, u_\tau y/\nu, \alpha) \tag{5}$$

where α is the trailing edge angle (if the flow is asymmetrical the ratio of the friction velocities on the upper and lower surfaces is an additional parameter). In the trailing edge region, there are large gradients in the x direction as well as in the y direction – that is, significant changes in mean velocity and other quantities occur in an x-wise distance of order ν/u_τ. Therefore, numerical solutions need an x step smaller than this, over a distance of at least $100\ \nu/u_\tau$, to achieve good accuracy – but this seems to be seldom realised.

The law of the wall for temperature is derivable by a simple extension and analogy of the analysis leading to Eq. (3). It implies that the turbulent Prandtl number is an absolute constant in the log. law region. Further reasoning, as plausible as that leading to Eq. (4), extends the law to the case where heat transfer rate $\dot{q}$, as well as shear stress, varies with y. The wall law for temperature is well confirmed by experiments in simple constant-stress, constant-heat-flux wall layers, but apparently (Kays & Crawford 1993) goes spectacularly wrong when τ varies with y even if $\dot{q}$ is nearly constant. This recent conclusion is disquieting, not only because the law of the wall for compressible flow relies on the assumption of constant turbulent Prandtl number but also because the temperature-law analysis is an innocuous extension of the velocity-law analysis, and failure of the former calls the latter into question – however well it may be supported by experiment.

4 Compressible flow

In the inner layer of a boundary layer in compressible flow, the shear stress is approximately equal to the surface value, but the density varies quite rapidly with distance from the surface (increasing as the temperature decreases with distance from the hot wall). The "Van Driest transformation" transforms inner-layer velocity profiles to fit the incompressible log. law. The transformation is, in effect, an integral of Eq. (4) with ρ as a function of y. Here T and hence ρ come from the assumption of a constant turbulent Prandtl number. In nearly-adiabatic flows below the hypersonic range, the assumption of constant total temperature gives a good enough prediction of density: this corresponds, strictly, to a Prandtl number of unity, but it is probable that small departures from the universal law of the wall for temperature (above) will have only a small effect on the velocity profile derived from the Van Driest transformation.

The Van Driest skin-friction formula is derived from the Van Driest transformation. Predictions of skin friction in compressible boundary layers (on flat plates in zero pressure gradient, say) are currently a subject of controversy, but there are certainly no experimental data that reliably invalidate the Van Driest skin-friction formula or the Van Driest transformation. This is probably the best justification for the extension of the law-of-the-wall analysis discussed in Sect. 3, but doubtless does little for the confidence of the determinedly subsonic.

In low-speed flow, the mean (streamwise) pressure gradient, as such, has almost no effect on turbulence (see Sect. 6). In compressible flow, streamwise pressure gradients change the density of fluid elements and can produce large changes in turbulence quantities, especially, of course, in flows through shock waves (e.g. Selig et al. 1989). Moreover, even pressure fluctuations which are not small compared with the mean pressure can affect turbulence. Specifically, if the Mach number based upon a typical fluctuating velocity and the local speed of sound is no longer small compared to unity, there may be significant dissipation of turbulent energy via dilatation fluctuations div $\mathbf{u}$, and significant correlations between fluctuations of pressure and of dilatation (Zeman 1990, Sarkar et al. 1991, Wilcox 1992). Measurements correlated by Birch & Eggers (1973) at NASA Langley show that the rate of spread of a turbulent mixing layer (in zero mean pressure gradient) starts to depend significantly on Mach number at Mach numbers close to unity. The data of Papamoschou & Roshko (1989) show even larger Mach-number dependence but other recent measurements are closer to the "Langley" curve. This Mach-number dependence apparently contradicts the well-known finding that the behaviour of compressible boundary layers can be quite well predicted by turbulence models that ignore compressibility effects (except of course that the right mean density must be used), at least for Mach numbers up to about 5. However, the typical turbulence intensity of a mixing layer is about five times that in a boundary layer, which implies that a mixing layer at $M=1$, based on the mean velocity difference across the layer, has the same ratio of velocity fluctuation to speed of sound (a.k.a. fluctuating Mach number) as a boundary layer at roughly $M=5$. A minor fallacy – strictly, a slight over-simplification – is that the behaviour of a two-stream mixing layer can be correlated on the "convective Mach number", a parameter related to the difference in speed between the two streams, irrespective of the individual stream Mach numbers. This seems to be not quite right, and causes confusion in comparisons of data.

Hypersonic boundary layers, $M>5$ approx., start to exhibit real-gas effects such as dissociation, and although these ought not to affect turbulence directly – given the mean density – they add to the general uncertainty of experimental data and hide any real effects of compressibility.

5 "Inactive" motion

The log-law analysis relies on the first-order hypothesis that u_τ, y and ν are the only relevant variables, which cannot be exactly and perfectly true. If the arguments that lead to Eq. (3) are applied to the turbulent motion they lead to results for the log-law region like $\overline{u^2}/u_\tau^2=\text{constant}$, whereas any flat-plate boundary-layer experiment shows a decrease with increasing y throughout the log-law region. This has led some people to regard the whole law-of-the-wall concept of local scaling as fallacious and its apparent success for the mean motion as fortuitous (an attitude supported by the fragility of the law of the wall for temperature, Sect. 3). Fortunately, this apparent discrepancy in the log-law analysis can be used to rescue the basic assumptions, by taking note of the so-called "inactive" motion (Townsend 1961, Bradshaw 1967). The concept is simple: the motion near the surface, even though it results mainly from eddies actually generated near the surface, is necessarily, affected by eddies in the outer part of the flow (i.e. those whose length scale is of the order of δ). Because the pressure fluctuation at a given point in a turbulent flow is derived from an integral of the governing Poisson equation over the whole of the flow, it follows that the eddies in the outer part of the boundary layer or pipe flow can produce pressure fluctuations which extend towards the surface and cause nominally-irrotational motion in the surface layer. An equivalent, alternative, explanation is the "splat" mechanism (a term introduced to the literature by Wood 1980) in which the large eddies in the outer flow are supposed to move towards the surface, to be reduced to rest by the normal-component "impermeability" condition at the wall, and to release their normal-component energy into the two tangential components u and w.

The "splat effect" motions, and the pressure fluctuations generated in the outer layer, have very long wavelengths in the x and z directions compared to the motions generated close to the surface. It follows from the continuity equation that the v-component velocity produced near the surface by outer-layer pressure fluctuations or large-eddy intrusions is of the order of y/λ times the u- or w-component velocity, where λ is the x- or z-component wavelength. Therefore the contribution of the "inactive" motion to the shear stress $-\rho\overline{uv}$ is small, of the order of y/λ – hence the name "inactive". Note that the "inactive" fluctuations are not entirely irrotational: the boundary condition $u=0$, $v=0$ at the surface results in the generation of a Stokes layer (see Sect. 6 on "slip velocity"). Even though "inactive" u-component fluctuations contribute significantly to $\overline{u^2}$, producing the anomalous y-dependence of $\overline{u^2}$ mentioned above, the effect on the mean law of the wall is very small. (A logarithm is a slowly changing function, so that fluctuations in u_τ have very little effect on the term $\ln(u_\tau y/v)$ in the log. law, and, therefore, the time-average velocity closely follows the log. law written with time-average u_τ.) The same arguments can be used to support the use of the log. law in unsteady-flow calculations at not-too-high amplitudes. The unsteady log. law must also be limited to not-too-high frequencies of unsteadiness: one would expect it to break down, at given y, at a frequency which was not small compared to the typical turbulence frequency u_τ/y. Very few unsteady-flow experiments reach frequencies high enough to disturb the log. law, the recent work of Tardu and Binder (1993) being an exception.

The contribution of the "inactive" fluctuations to the power spectra of u and w at low wave numbers ($\approx$low frequencies: wave number$=2\pi/$[wavelength]) is considerable, resulting in very large differences between the measured spectra in typical turbulent flows and those predicted by inner-layer analysis. The latter predicts that the wave-number spectral density should scale on u_τ and y, and that the wave number k should appear as ky (since we have neglected v, this applies only for $u_\tau y/v>30$ and at wave numbers small compared with the viscous limit, but neither restriction concerns us here). In practice, there is an apparent Reynolds-number effect at given $u_\tau y/v$: strictly it is a y/δ effect, but $y/\delta \equiv (u_\tau y/v)/(u_\tau\delta/v)$.

In the atmospheric boundary layer, which is of the order of 1 km thick, the inactive motion effects on spectra measured at the standard height of 10 m are very large, and in particular the u-component spectrum follows a $-5/3$ power law down to very low wave numbers. This phenomenon, which is present, but less spectacular, in laboratory boundary layers, has been the cause of a large amount of confusion, controversy and difficulty, because the classical Kolmogorov scaling indicates that the spectrum should vary as $k^{-5/3}$ only for wave numbers large compared to those of the energy-containing eddies. In the context of the atmospheric boundary layer at a height of 10 m this means wavelengths much smaller than 10 m. The fact that the experimentally-observed spectrum follows the $-5/3$ law down to wave numbers far lower than could be expected from the arguments of inner-layer scaling and the Kolmogorov universal-equilibrium hypothesis is one of the most difficult "fallacies" in turbulent flow: it is of course a case of post hoc ergo propter hoc.

In summary, the qualitative idea of "inactive motion" explains both the apparent failure of inner-layer scaling and the unexpected success of the $-5/3$ law.

6 "Slip velocity"

Several difficulties or misconceptions about turbulent flows over walls can be cleared up if we recall that the very thin viscous wall region $u_\tau y/v<30$ really produces what might be called a "slip velocity" between the fully-turbulent flow and the surface. As well as the obvious example of Reynolds-number (and Peclet-number) effects, the slip-velocity concept helps to understand, or at least to correlate, the effects of pressure gradient, unsteadiness and surface roughness.

6.1 Effects of Reynolds number and Peclet number (viscosity and conductivity)

If the Reynolds number of a turbulent flow – based on total thickness and, say, the square root of the maximum shear stress or turbulent energy – is large, classical (e.g. Kolmogorov) theory suggests that the details of the turbulent motion should be independent of Reynolds number, except for the very smallest eddies which are responsible for viscous dissipation of turbulent kinetic energy into thermal internal energy. In this respect at least, classical theory seems to be correct, and there is no significant evidence to refute it. If the Reynolds number of a given turbulent eddy, made with its typical velocity fluctuation and its typical length scale, is large, there is no reason why viscous effects on the eddy should be significant. (This statement should strictly be phrased in statistical terms!) In a pipe flow, half the mean-square u-component intensity near the centre-line comes from wavelengths larger than the pipe diameter, so the "eddy Reynolds number" of the main energy-containing eddies is of the same order as the mean Reynolds number defined at the start of the paragraph, and we can use the former for simplicity. The "energy cascade" process of Kolmogorov theory, attributable to random vortex stretching, implies that turbulent energy is transferred from energetic eddies of low wave number (i.e. large Reynolds number) to weak eddies of high wave number (small Reynolds number), and although back-scatter transfer from small eddies to large can occur intermittently, the time-average transfer of energy is from the large eddies to the small and there seems to be no significant "back scatter" of viscous effects.

Near a solid surface ($y^+ < 30$ say) the largest eddies, whose wavelength is roughly equal to y, are no longer very large compared to the smallest eddies (the smallest-eddy length scale, Kolmogorov's η or l_k, is about $0.06y$ at $y^+ = 30$), so the energy-containing eddies – which also carry the shear stress – start to depend on viscosity. (Also, and slightly differently, the mean velocity gradient becomes so large that viscous shear stress is a significant fraction of the total shear stress.) Therefore, viewed from the outer part of the flow, there is a viscosity-dependent region near the wall and so the velocity difference between the surface and, say, $y/\delta = 0.1$, depends on Reynolds number. Viewed from the outer part of the flow, there is a Reynolds-number-dependent "slip velocity" at (strictly near) the surface.

In a free shear layer (wake, jet, mixing layer . . .) there is no true viscous effect unless the Reynolds number is so low that turbulence can only just exist. However, free shear layers can be quite strongly dependent on the initial conditions, for long distances downstream, and since the initial conditions frequently do depend on Reynolds number there is a "pseudo-viscous" effect.

A corollary of the negligibility of viscosity as part of turbulent transport of momentum is the negligibility of conductivity in the transport of heat or mass by turbulence. Briefly (again) the "turbulent Prandtl number" is independent of the molecular Prandtl number unless the Reynolds number based on eddy velocity scale and eddy length scale, i.e. $u_\tau y/\nu$, is small.

6.2 *Effect of pressure gradient*

Another of the standard incomprehensions about turbulent flow is the effect of (streamwise) mean pressure gradient on the turbulence as such: recall that we are considering only incompressible flow. It arises partly because experimenters tend to normalize their turbulence measurements by the local mean velocity. In adverse pressure gradient, say, the mean velocity decreases with increasing x so the normalized turbulence intensities, shear stress etc. increase. However it can easily be shown that absolute turbulence properties on a given streamline are only slightly affected by pressure gradient:-

The Reynolds-stress transport equations do not contain the mean pressure (they contain correlations between the pressure fluctuation and instantaneous rate of strain, but pressure fluctuations have no connection whatsoever with the mean pressure). Also, the z-component mean vorticity $\partial V/\partial x - \partial U/\partial y$ is unaffected by pressure gradient, and if we assume that the boundary layer approximation is valid this means that $\partial U/\partial y$ is unaffected, even though the pressure change leads to a change in velocity all through the shear layer and thus may change δ significantly. Alternatively, recall that a static-pressure gradient does not affect the total pressure P (on a given streamline) directly. In the simple case of two-dimensional flow, therefore, $\partial P/\partial\psi$, where ψ is the stream function, is unaffected, and if $\partial p/\partial y$ is negligible, as required by the boundary-layer approximation, a little algebra shows that $\partial U/\partial y$ is unaffected. At the surface, where the total pressure is equal to the static pressure, there is a change in P and $\partial U/\partial y$, produced of course by viscous stresses. The "internal layer", in which the total pressure and mean vorticity rise to their unaffected profiles, gradually spreads out from the surface, but outside this the static-pressure gradient has no effect except to reduce the mean velocity and thus thicken the boundary layer. This result applies to laminar or turbulent boundary layers (or other wall flows such as those in tapered ducts). In summary, the initial effect of pressure gradient is confined to the "slip velocity" at the wall.

Mean pressure gradients do have some effects on the turbulent motion. Adverse pressure gradient stretches eddies in the y direction, because the shear layer thickens: however, the area, in side view, of a given eddy or fluid element is unaltered, and so if we suppose that the length scale of an eddy is just the square root of its area in side view, or the cube root of its volume, the length scale is unaltered. (This is admittedly a crude argument.) Of the terms in the Reynolds-stress transport equations, the only ones directly affected are the y-component diffusion terms, which are the derivatives of various triple products, etc., with respect to y. If the triple product on a given streamline is unaffected but the streamlines diverge in the $x-y$ plane because of the adverse pressure gradient, the y-wise derivative is reduced.

6.3 *Unsteadiness*

The effect of unsteadiness can be understood in the same way as that of pressure gradient – of course, unsteadiness is usually forced by a streamwise pressure gradient. In the case of unsteady laminar flow the internal layer is called a Stokes layer. There are close correspondences in laminar flow between an infinite oscillating plate in still air and flow over an infinite stationary surface driven by an oscillating pressure gradient, and the qualitative correspondence carries over to turbulent flow. If the pressure gradient is strong enough to cause separation (however defined), the internal layer is carried into the outer part of the flow and the "slip velocity" concept breaks down, as it would in steady separation.

6.4 Surface roughness and perturbations of the viscous sublayer

Boundary layers with heat transfer in liquids may have significant changes in viscosity across the viscous sublayer, leading to a change in the additive constant C in the logarithmic law (based by convention on the viscosity at the wall) or equivalently a change in slip velocity. The effect of most kinds of surface roughness of length scale k can be correlated as a reduction in C or in slip velocity which is a function of $u_\tau k/\nu$ that depends on roughness geometry. The effect of drag-reducing riblets can be similarly correlated as an increase in C. (The same applies to long-chain polymers, whose effect is confined to the viscous sublayer where the length and time scales of the turbulence are comparable to those of the molecules.) Another occasion where a change of boundary condition affects the flow only in an "internal layer" is the flow downstream of a change in surface roughness. This is comparatively rare in aerodynamics but an important case in meteorology where, for example, air can flow from the "smooth" ocean to the land and undergo a change of apparent surface roughness. Indeed, the internal-layer concept was first proposed to describe this case. As the surface boundary condition changes, the additive constant C in the logarithmic law for a smooth surface is replaced by the appropriate value for a rough surface. The effect of this change in surface boundary condition spreads outward from the surface at an angle of the order of rms u/U, i.e. of the order of 3%, so that the rate of contamination of outer-layer turbulence by inner-layer changes is no greater than about 1 or 2 degrees. Since the pressure gradient is nominally zero there is no streamline divergence above the internal layer, although the change in velocity in the internal layer produces a vertical displacement of the outer flow (upwards, in the case of a smooth-to-rough change where the flow in the internal layer is retarded).

7 Spectra and convection velocity

Classical turbulence theory aims to predict all the statistical properties, not simply the Reynolds stresses. In particular it deals with the statistical distribution of eddy sizes. It is usually formulated in terms of wave-number spectra, wave number k being a vector with the direction of wavelength and the magnitude of 2π/wavelength. (The alternative is two-point spatial correlations, which are less convenient mathematically.) Wave-number spectra are the Fourier transforms of the two-point correlations, but a full description requires correlations for all magnitudes and directions of the distance between the two points, or spectra for all magnitudes and directions of the wave number). In most experiments only frequency spectra, and a few correlations along the coordinate axes, are measured.

Apart from the Kolmogorov theory for the high-wave-number part of the spectrum (discussed below), analytical results are rare. Inner-layer scaling merely states that the spectral density $\phi_{ij}(k)$, whose integral over all k is $\overline{u_i u_j}$, scales like

$$\frac{\phi_{ij}(k)}{u_\tau^2 y} = f(ky, u_\tau y/\nu) \tag{6}$$

where k is any component of k and the parameter $u_\tau y/\nu$ simply controls the viscous cutoff at high k. If the larger-scale (lower k) motion consists of so-called attached eddies, with geometrically similar shapes but a range of sizes, then the motion at different values of y is similar, y is no longer a relevant length scale, and so, below the viscous-dependent rage of k, we have

$$\phi_{ij}(k) = u_\tau^2/k. \tag{7}$$

This does not seem to be well supported by measurements, except insofar that every spectrum must have a k^{-1} region somewhere. A similar argument can be used on the surface pressure fluctuation, where y is not a meaningful variable, and again a k^{-1} spectrum is predicted. Again data are not encouraging, but the surface pressure fluctuation receives contributions from eddies at different values of y, moving downstream at different speeds, and conversion from a measured frequency spectrum to the wave-number spectrum discussed in the theory is not straightforward.

This is the best place to comment on the definition of "frequency" in turbulence. The frequency seen by an observer moving with the mean flow is (velocity scale of turbulence) $\div$ (length scale of turbulence) – for example u_τ/y in the inner layer – but the frequency seen by a fixed observer is approximately (MEAN velocity) $\div$ (length scale of turbulence) and is usually much larger. The reciprocal of the moving-observer frequency is sometimes called the "eddy turnover time": this is of course an order-of-magnitude concept. A related difficulty is the status of time derivatives: all transport equations in fluid flow, including the Navier–Stokes equations, have the operator

$$\frac{\partial}{\partial t} + u_l \frac{\partial}{\partial x_l} \tag{8}$$

on the left-hand side. It is called the substantial derivative, or the transport operator, and it is the rate of change with time seen by a fluid element. The relative size of the temporal and spatial derivatives depends on the velocity of the observer but the sum of the derivatives does not.

The fixed-observer frequency is used to deduce x-component wave-number spectra from frequency spectra, using Taylor's hypothesis that the speed at which the turbulence pattern moves past the observer (its "convection velocity") is closely equal to the mean velocity. It is qualitatively obvious that this will only work well if the mean velocity is large compared to the velocity scale of turbulence, so that an eddy is carried past the measurement point in a time very much less than its turnover time. A more precise analysis is possible:-

There are various definitions of the actual "convection velocity" of turbulence: most are in effect phase velocities and therefore not ideal for considering convection of turbulent kinetic energy or Reynolds stress. A plausible definition of a group (energy-transport) velocity comes from considering the streamwise (say, x-wise) "diffusion" of turbulent energy (transport of the turbulent energy by the turbulence): the energy flux rate, whose x derivative appears in the turbulent energy equation, is $\overline{p'u}/\rho+(\overline{u^3}+\overline{uv^2}+\overline{uw^2})/2$. Rates of energy flux due to pressure fluctuations seem to be small – except perhaps near the free-stream edge of a turbulent flow where pressure fluctuations drive an "irrotational" motion which extends outside the vortical region – and are certainly not measurable, which is some justification for neglecting them. Doing this, and writing $\overline{q^2}$ for $\overline{u^2}+\overline{v^2}+\overline{w^2}$, (so that the turbulent kinetic energy is $\overline{q^2}/2$), the above energy flux rate can be written as $(\overline{u^3}+\overline{uv^2}+\overline{uw^2})/\overline{q^2}$. We can define the transport velocity of turbulent energy as this flux rate divided by the turbulent energy. The largest contribution to the numerator is $\overline{u^3}/2$ – though the others are not negligible – so the transport velocity is of order $\sqrt{(\overline{q^2})}\times S_u$, where S_u is the skewness of u. Now S_u lies in the range ± 1 approx. over most of a boundary layer, so we can finally say that the x-component transport velocity of turbulent energy is not more than a few times $\sqrt{(\overline{q^2})}$. Since this is the difference between the group velocity of the turbulence and the mean velocity, we see that the difference is a small percentage of the mean velocity in flows with low turbulence intensity, such as boundary layers. This quantitatively justifies the use of Taylor's hypothesis in such flows and of course allows an estimate of its inaccuracy in highly-turbulent flows.

Differences between convection velocity and mean velocity are large near the free-stream edges of mixing layers and jets. In these regions the irrotational motion, induced by pressure fluctuations generated in the high-intensity region of the flow near the inflexion point(s) in the velocity profile, is strong compared to the true (vorticity-carrying) turbulence, and its convection velocity is necessarily close to the mean velocity in the high-intensity region. The rotational motion (vorticity pattern) seems to travel at a speed close to the local mean velocity, as predicted by the above analysis (intensities near the outer edge of a jet are not large). In terms of the above analysis, the streamwise transport velocity of the vorticity pattern is still dominated by the triple-product terms, while $\overline{p'u}/\rho$ determines the transport velocity of irrotational motion.

The de Havilland Comet I jet airliner had four engines, buried in the wing roots. The designers carefully arranged that the jets themselves would clear the fuselage, but forgot the "near field" pressure fluctuations – far more intense than the jet noise – that drive the irrotational motion. The pressure patterns, travelling at the above-mentioned convection velocity, produced fluctuating stresses at the fixed-observer frequency in the aircraft skin, which led to fatigue of the aluminium.

Later marks of Comet had the engines toed out.

Misconceptions about turbulence can be expensive!

8 The microscale and the Kolmogorov theory

Frequently, the Taylor "microscale" is used as a length scale in discussions of wave-number (or frequency) spectra.

The microscale λ is a hybrid scale of turbulence. It is usually defined by

$$\lambda^2=\overline{u^2}/\overline{(\partial u/\partial x)^2} \tag{9}$$

(other definitions with different choices of velocity component or gradient direction occasionally appear). This is an equation whose numerator is a property of the larger, energy-containing eddies, but whose denominator is a property of the dissipating eddies (if the dissipating eddies are statistically isotropic the dissipation rate is $15\nu\overline{(\partial u/\partial x)^2}$). For this reason it is a misconception to regard the microscale as the length scale of any particular group of eddies: it actually lies closer to the length scale of the dissipating eddies than that of the energy-containing eddies. Professor Charles Meneveau of Johns Hopkins University (private communication) points out that the microscale is the typical distance through which the energy-containing eddies convects the dissipating eddies during one time scale of the latter: this apparently far-fetched interpretation is actually relevant to Kraichnan's direct-interaction theory for the spectrum in homogeneous turbulence (Leslie 1973) which needed modification to account for the scrambling effect of this convection. The Reynolds number based on the microscale and the root-mean-square turbulence intensity, $\lambda(\overline{u^2})^{1/2}/\nu$, however, has a more understandable meaning. If the Reynolds number is high enough for the dissipation to be equated to the isotropic formula, the microscale Reynolds number is proportional to the square root of an "eddy" Reynolds number for the energy-containing motion, based on the rms turbulence intensity and the dissipation length scale $L\equiv(\overline{u^2})^{3/2}/\varepsilon$. Of course, this does not give the microscale the status of Eddy Length Scale post hoc.

The effect of mean strain rate on the small-scale motion is sometimes correlated in terms of Sk/ε where S is the strain rate. However this parameter is closely related to the ratio of TKE production to dissipation, which is strictly a parameter of the energy-containing motion. The right parameter is the ratio of S to a typical strain rate of the smallest eddies, $\sqrt{\varepsilon/v}$.

It is important to notice that the "dissipation" in the definition of L is in fact the rate of transfer of turbulent kinetic energy from the large eddies to the smallest eddies which is, by all prevailing turbulence theories, supposed to be a property of the large eddies rather than the smallest eddies. The smallest eddies simply rearrange themselves to dissipate the energy handed down to them. If the turbulence is changing slowly with time (or streamwise distance) then, of course, the rate of transfer from the large eddies to the smallest eddies is equal to the rate at which energy is being dissipated by the smallest eddies, but this is not formally an equality because the "cascade" process is not instantaneous. In rapidly-changing turbulent flows the "equilibrium" arguments fail, and the rate of transfer from the energy-containing eddies to the dissipating eddies is not equal to the rate at which energy is being transferred from the dissipating eddies to heat. This restriction on Kolmogorov's "universal equilibrium" theory, which we used in Sect. 6.1, is too often forgotten.

Another restriction of the Kolmogorov theory is that, of course, energy which is transported in the y direction by turbulent "diffusion" will be generated at small y, but dissipated at large y where the statistical properties are different. In particular, in flows with a free-stream boundary, energy is generated in regions of large mean shear and then transported in the positive y direction to regions of zero or negligible mean shear before being dissipated. The energy transfer through the inertial subrange at the second location is likely to be intermediate between the dissipation rates at the two locations.

Nevertheless results from a large number of experiments on turbulent shear layers have recently been analysed (Kuznetsov et al. 1992) to show that Kolmogorov scaling works remarkably well when adjusted for the intermittency factor γ (the fraction of time for which the flow at a given location is turbulent). In an intermittent region, the average of any turbulence quantity within the turbulent part of the flow is $1/\gamma$ times the conventional average over all time. For example the conventional-average spectral density and the dissipation ε must both be multiplied by $1/\gamma$. However the Kolmogorov "$-5/3$" law for the spectral density in the so-called inertial subrange contains $\varepsilon^{2/3}$ so that, formally, there is a spare factor of $\gamma^{1/3}$ and we certainly do not expect the Kolmogorov law to hold if written with conventional-average quantities. The data analysis of Kuznetsov et al. shows that the Kolmogorov formula still works for a wide range of intermittent flows when written for the turbulent part of the flow, i.e. taking account of the "spare factor", and using the dissipation rate at the local value of y. Since the formula strictly applies only to nearly-homogeneous turbulence, and an intermittent region, almost by definition, contains only one large eddy at a time, this result is a surprising testimonial to the robustness of the Kolmogorov theory. Needless to say, the usual cautions about post hoc apply.

Measurements in the one-metre-thick roof boundary layer of the NASA Ames Research Center 80 ft × 120 ft. tunnel by Saddoughi & Veeravalli (paper to appear in J. Fluid Mech.) showed that a microscale Reynolds number (Sect. 8) of more than 1000 is needed to obtain a convincing locally-isotropic inertial subrange, but that the Kolmogorov theory is indeed an accurate description of high-*Re* turbulence.

9 Turbulence modelling

9.1 *Normal pressure gradients*

An incomprehension entirely unrelated to turbulence, which nevertheless causes confusion in tests of turbulence models, is the effect of normal pressure gradient on boundary layers and other shear layers. If the shear layer obeys the boundary layer approximation then, by definition, the pressure gradient in the y direction is negligibly small. However, if in a real flow the normal pressure gradient is not negligible, there will be a velocity gradient $\partial U/\partial y$ even in the external stream (where the total pressure is constant) and the velocity gradient will, in principle, lead to extra production of turbulence via the product of mean velocity gradient and turbulent shear stress. Of course, the same effect occurs within the shear layer $y<\delta$, but is less easily identified. Therefore, even if a turbulence model produces exactly correct predictions of the shear stress – given the mean velocity profile as input – it will not give acceptable results in the case where normal pressure gradients affect the mean velocity gradient. (Recall that the boundary-layer momentum equation can be written as $dP/dx = d\tau/dy$.) This is probably a much more important reason for inaccuracy of predictions based on the boundary layer approximation in rapidly-growing flows near separation than the often-quoted presence of significant normal-stress gradients. Incidentally one of the most widely-propagated fallacies in fluid dynamics is based on Prandtl's oft-reproduced sketch of (laminar) separation from his 1904 paper on boundary layers: the separation streamline is shown leaving the surface at about 30 deg., but unfortunately the annotation "Vergrößert'" (enlarged) on the vertical scale is often omitted from the reproduction. Boundary layers usually leave the surface rather gradually; if the rate of growth of the displacement thickness suddenly increased, the outer inviscid flow would impose an increased adverse pressure gradient upstream of separation, thus moving the separation point upstream to smooth out the discontinuity.

9.2 Universality

Perhaps the biggest fallacy about turbulence is that it can be reliably described (statistically) by a system of equations which is far easier to solve than the full time-dependent three-dimensional Navier–Stokes equations. Of course the question is what is meant by "reliably", and even if one makes generous estimates of required engineering accuracy and requires predictions only of the Reynolds stresses, the likelihood is that a simplified model of turbulence will be significantly less accurate, or significantly less widely applicable, than the Navier–Stokes equations themselves – i.e. it will not be "universal".

Irrespective of the use to which a Reynolds-stress model will be put, lack of universality may interfere with its calibration. For example, it is customary to fix one of the coefficients in the model dissipation-transport equation so that the model reproduces the decay of grid turbulence accurately. This involves the assumption that the model is valid in grid turbulence as well as in the flows for which it is intended – presumably shear layers, which have a very different structure from grid turbulence.

It is becoming more and more probable that really reliable turbulence models are likely to be so long in development that large-eddy simulations (from which, of course, all required statistics can be derived) will arrive at their maturity first. The late Stan Corrsin (1973) described the process of turbulence modelling as a "trek to determinacy", while Hans Liepmann described it as "resting on a shaky foundation". Certainly, over the last twenty years the rate of progress in turbulence modelling has been pretty small compared to the rate of progress in development of digital computers, and the consequent increase in Reynolds-number range and geometrical complexity attainable by simulations. Very possibly, Reynolds-averaged turbulence models will go the same way as viscous-inviscid interaction, which has been largely superseded by Navier–Stokes, or parabolized-Navier–Stokes, calculations. Until recently, most work has concentrated on "complete" simulations, covering the whole range of eddy sizes, while large-eddy simulations, which alone offer the prospect of predictions at high Reynolds numbers, have been somewhat neglected. However the recent breakthrough in sub-grid-scale modeling for large-eddy simulations (Germano 1992) has led to a resurgence of enthusiasm for LES: LES calculations of the flow over a backward-facing step by Akselvoll and Moin (1993) agreed, within an acceptably small tolerance, with the full simulation results of Le and Moin (1992) and cost only about 1/50 as much as the full simulations.

9.3 Eddy viscosity and gradient transport

Turbulence models which invoke an eddy viscosity (of whatever type) necessarily produce pseudo-laminar solutions with the stresses closely linked to the mean-flow gradients: they may be well-behaved but they are not usually very accurate away from the flows for which they have been calibrated. Turbulence models based on term-by-term modelling of the Reynolds-stress transport equations produce solutions which may be accurate in some cases, but are liable to fail rather badly in other cases: that is, they are "ill-behaved" in a way that eddy-viscosity methods are not.

It may be this "reliable inaccuracy", rather than the larger computer resources needed for Reynolds-stress transport models, which has led to two-equation (e.g. k, ε) or even one-equation methods being the industry standard. The one-equation Baldwin–Lomax (1978) turbulence model has been extended – by others – far beyond the domain intended by its originators, simply because it has the virtue of almost never breaking down computationally!

It has, of course, often been said that it is just as unreliable and unrealistic to define an eddy viscosity entirely in terms of turbulence properties (as in the k, ε method) as to define it entirely in terms of mean-flow properties as in the Baldwin–Lomax (1978), Baldwin–Barth (1990), or Spalart–Allmaras (1992) methods. Eddy viscosity is the ratio of a turbulence quantity (i.e. a Reynolds-stress) to a mean-flow quantity (i.e. a rate of strain or velocity gradient), so, like the microscale, it is a hybrid quantity.

Minor fallacies in turbulence modelling abound, but misuse of gradient-transport hypotheses is probably responsible for more than its fair share. One of the most spectacular was the use many years ago, by authors I will not identify, of the gradient-transport approximation for diffusion of turbulent energy by pressure fluctuations. In terms of classical physics, anything less likely than pressure diffusion to obey a gradient-transport approximation could scarcely be imagined. A fallacy which has, in charity, to be regarded as a deliberate approximation, is the use – even in Reynolds-stress transport models – of the eddy-diffusivity (gradient-transport) approximation for the turbulent transport terms. It appears that most of the turbulent transport of Reynolds stress is provided by triple products of velocity fluctuations, rather than by the pressure diffusion just mentioned, and therefore a gradient-transport approximation is not so obviously unphysical. However it has recently been shown (Huang et al. 1993) that the presence of density gradients in the turbulent transport term of the conventionally-modeled equation for dissipation transport (Sect. 9.4), causes large discrepancies between model solutions and the expected "law of the wall" solution $\varepsilon=(-\overline{uv})^{3/2}/\kappa y$. Whatever the status of the law of the wall, the dissipation-transport equation model is too fragile to undermine it, and the conclusion is that the usual modeling of the dissipation-transport equation is inadequate.

9.4 The dissipation-transport equation

Most turbulence models, whether relying on an eddy viscosity or on the Reynolds-stress transport equations, use the dissipation-transport equation to provide a length scale or time scale of the turbulent flow. Strictly, the length scale or time scale required is that of the energy-containing Reynolds-stress-bearing eddies, not that associated with the dissipating eddies as such, and so two questions arise. One is whether the rate of dissipation is adequately equal to the rate of energy transfer from the large eddies (which clearly, is the quantity that we really want to model); the other is whether, if we really pretend to be using the dissipation transport equation – all of whose terms depend on the statistics of the smallest eddies – we can logically model those terms by using the scales of the larger, energy-containing eddies. I think it is inescapable that current models of the so-called dissipation transport equation, which certainly do parameterize the terms as functions of the large-eddy scales, start out with the dissipation-transport equation as such and end up with a totally-empirical transport equation for the energy transfer rate. In other words, the relation between the "dissipation" transport models and the exact transport equation for turbulent energy dissipation is so tenuous as not to need consideration. Unfortunately, even Reynolds-stress transport models usually employ this suspect dissipation-transport equation to provide a length scale, and this is undoubtedly one of the reasons why Reynolds-stress transport models have not outstripped two-equation models. A less-used alternative to the ε equation is the ω equation (admitted to be totally empirical). ω is nominally proportional to ε/k where k is the turbulent kinetic energy, but conversion from one to the other (in either direction) produces the interesting result that the turbulent transport terms in the transport equation for the first quantity (the integral of transport terms over the flow volume being by definition zero) convert to a transport term plus a "source" term in the equation for the second quantity. There is increasing evidence that using ω to provide a length scale gives better results than using ε: if there is a reason other than more judicious choices of empirical coefficients, it must lie in the above-mentioned source term.

9.5 Invariance

One of the customary requirements of a turbulence model is that it should be "invariant" (with respect to translation or rotation of axes). The boundary layer (thin-shear layer) equations are not invariant: it is therefore quite unrealistic to expect a shear-layer model to be totally invariant, and it is perfectly realistic to suppose that the direction normal to the shear-layer (y) is a special direction. There seems to be no reason why a turbulence model should not, given an identifiable "special direction" in a shear layer, use that special direction for orientation of its empirical constants and functions. Even though equations (such as the Navier–Stokes equations or the time-average Reynolds equations) may be invariant, the boundary conditions for which they are to be satisfied certainly are not invariant (almost by definition). Therefore, the solutions of the exact, or approximate, equations of motion of turbulent flow cannot be expected to be invariant with respect to translation or rotation. From this it is a rather small step to argue that the empirical constants or functions in these model equations should, again, be released from invariance requirements.

9.6 Local modelling of pressure-fluctuation terms

The mean products of fluctuating pressure and fluctuating rates of strain that act as redistribution terms in the Reynolds-stress transport equations represent, very crudely speaking, the effect of eddy collisions in making the principal Reynolds stresses more nearly equal – that is, making the turbulence more nearly isotropic (statistically). The shear stress in isotropic turbulence is zero, so the effect of the pressure-strain terms on the shear stress, and their modelling, is of great interest.

Pressure fluctuations within a turbulent flow are one of the Great Unmeasurables: they are of the order of ρu^2 and so, unfortunately, are the pressure fluctuations induced on a static-pressure probe by the velocity field. That is, the signal-to-noise ratio is of the order of one. To say that signals cannot be educed even with $S/N = O(1)$ is itself a fallacy, but in this case the attempts made to do so have not met with general acceptance. Pressure fluctuations can be extracted from simulations, but these are confined to low Reynolds number.

An equation for the pressure (mean and fluctuating) can be obtained by taking the divergence of the Navier–Stokes equations. It is a Poisson equation, and it is necessary in turbulence modelling to consider the different terms on the right-hand side separately, by writing a Poisson equation for each and adding the solutions to get the pressure. One such is the equation for the "rapid" pressure, which for a two-dimensional boundary-layer flow is

$$\frac{\nabla^2 p}{\rho} = -2\frac{\partial U}{\partial y}\frac{\partial v}{\partial x}. \tag{10}$$

The "rapid" pressure is so called because it responds immediately to a change in the mean flow, as represented by $\partial U/\partial y$. To regard this apparently-surprising fact as physically meaningful is a misconception: it is just a result of the way we take averages, and, obviously, the turbulence at a given instant does not know what the mean flow is. A highly symbolic solution of the equation is

$$\frac{p'}{\rho}=-2\nabla^{-2}\frac{\partial U}{\partial y}\frac{\partial v}{\partial x} \tag{11}$$

where ∇^{-2} is a weighted integral over the whole flow volume. In other words, the "rapid" pressure at a given point, and its contribution to the pressure-strain terms at that point, depend on conditions for a distance of several typical eddy length scales around that point – i.e. they are "non-local". The same non-locality accounts for the presence of irrotational velocity fluctuations outside the turbulent motion.

Almost all current stress-transport turbulence models, with the exception of that of Durbin (1993), model the pressure-strain terms and other pressure-velocity correlations entirely as functions of local quantities. (All the other terms in the Reynolds-stress transport equations are genuinely local quantities.) This is equivalent to replacing Eq. (11) by

$$\frac{p'^B}{\rho}=-2\frac{\partial U}{\partial y}\nabla^{-2}\frac{\partial v}{\partial x} \tag{12}$$

– that is, evaluating $\partial U/\partial y$ at the position where p' is required and volume-integrating only $\partial v/\partial x$.

Bradshaw et al. (1987) analyzed the behavior of existing models for the pressure-strain terms, using simulation data in a duct flow to evaluate the terms directly. The results, surprisingly, suggest that the difference between the exact pressure-strain terms, using p' from Eq. (11), and the approximate results, using p'^B from Eq. (12), is negligibly small (or, at least, small enough to be hidden in the empirical coefficient in the pressure-strain model) except in the viscous wall region. Within the viscous wall region, the difference between the true pressure fluctuation p' and the approximate pressure fluctuation p'^B is not only very large but eccentrically behaved. It is not suggested that viscous effects, arising from the $u=0$ boundary condition at the surface, are directly to blame: it is much more likely that the effects of the $v=0$ boundary condition are mainly responsible, but it is surprising that these effects should be small outside the viscous wall region. A final possibility is that the changes in turbulence structure with $u_\tau y/\nu$ in the viscous wall region are so large as to invalidate local models.

This suggests not only that standard pressure-strain models are grossly inaccurate in the viscous wall region, but also that any extension of a standard turbulence model into the viscous wall region will be similarly inaccurate. This inaccuracy can be camouflaged by the insertion of "low-Reynolds-number" functions, nominally functions of the wall distance $u_\tau y/\nu$. Obviously, if the real flow scales with $u_\tau y/\nu$, this simple procedure suffices, but if the flow approaches, or goes beyond, separation then inner-layer scaling – and presumably "low-Reynolds-number" models – break down, even if $u_\tau y/\nu$ is replaced by the guaranteed-real quantity $k^{1/2}y/\nu$.

9.7 Orderly (coherent) structures

The early concept of turbulence was that of rather small ill-organized eddies: in the 1950s Townsend first put forward the idea that shear layers are dominated by large eddies, with a well-defined statistical-average shape. His original concept was that the large eddies were weak, most of the turbulent kinetic energy being contained in the smaller-scale motion: this would imply that the large eddies carry little shear stress and therefore have little effect on the production of turbulent energy, but Townsend (1956) hypothesized that they controlled the smaller-scale motion and thus determined the shear stress. In fact, as Grant (1958) soon found, the large eddies do carry a significant percentage – though not a majority – of the turbulent energy. This invalidated Townsend's hypothesis and alternative turbulence models of that era were far too crude to accommodate information about eddy structure.

Brown and Roshko (1974) showed that the fluctuating motion in a laboratory mixing layer consisted mainly of long-lived spanwise vortex rolls, and this started a rather intense search for simply-shaped and/or long-lived eddies in all kinds of turbulent flow. Many of the earlier experiments were carried out in mixing layers at very low Reynolds number, where the flow had not reached its asymptotic turbulent state, and more recent work has revealed a much more complicated large-eddy structure in mixing layers. Basically, the spanwise vortex rolls which are the instability eigenmodes of a laminar mixing layer can maintain themselves in the presence of fully-three-dimensional background turbulence (for what it is worth, the Reynolds number of a turbulent mixing layer based on eddy viscosity is larger than the critical Reynolds number of a laminar mixing layer). The reason is the very high amplification rate of inflexion-point instability. The wake shows some evidence of two-dimensional orderly structure but boundary layers, which do not have inflexional profiles except in strong adverse pressure gradient, seem to have three-dimensional large eddies whose ideal form is a horseshoe vortex rather than a 2D vortex roll (Robinson 1991). The fallacy about "coherent structure" is simply that the timescale of the energy-containing eddies is bound to be (turbulent kinetic energy)/(dissipation rate) – dissipation rate being of the same order as production rate – and so any group of eddies that has a much longer lifetime than this (which is presumably what is meant by "coherent") cannot carry a large fraction of the turbulent energy. Also, if large eddies are "orderly" and uniformly-spaced then by definition their spatial correlations are oscillatory and decay slowly at large separation – but slowly-decaying correlations are rarely found in experiment.

The search for orderly structure has produced large quantities of spin-off knowledge of turbulence behaviour, but seems to have had almost no impact on modeling. The concept of well-organized large eddies carrying quasi-random background turbulence is difficult to incorporate in Reynolds-averaged models. Hong and Murthy (1986) presented a model which amounts to (very) large-eddy simulation with one large-eddy mode but computing times are obviously very long.

10 Chaos

Chaos has been one of the buzzwords in applied mathematics in recent years, and turbulence is often cited as the supreme example. The complication of turbulent motion, with its broad spectrum of wavelengths, is far greater than that of the "chaotic" solutions of some low-order systems of coupled ordinary differential equations. Analysis of simulation data (Keefe 1990) suggests that the dimension of the turbulence attractor (roughly, the number of modes or "degrees of freedom" needed to represent the turbulent motion) is several hundreds at least, even at the lowest Reynolds number at which turbulence can exist. The upper bound on the dimension is, roughly, the number of totally-arbitrary modes needed to represent the motion. Now since direct-simulation calculations need, typically, $128^3 \approx 2 \times 10^6$ Fourier modes or finite-difference points for flows at a very modest laboratory-scale Reynolds number, we can take the upper bound of the attractor dimension as being of this order: for the barely-turbulent flow in a duct at $Re = 2000$, $32^3 \approx 30000$ might do. Large-eddy simulations need fewer points: 128^3 might do for any Reynolds number, at least if the viscous wall region did not have to be resolved. These are all impracticably large estimates of the attractor dimension.

However, several authors have based their work on the classically incorrect syllogism "Solutions of some equations with few degrees of freedom yield complicated behavior: turbulence has complicated behavior: therefore turbulence may be represented by the solution of equations with few degrees of freedom". This would have pained Aristotle (fl. 350 B.C.). The conclusion of the syllogism was of course postulated many years B.C. (before chaos), and a great deal of brain power has been applied to prove it – i.e. to produce a usably small set of modes to describe turbulence – but without great success: the most ambitious efforts (Hong and Murthy 1986) require an amount of computing time which is not much less than that of a large-eddy simulation.

The concepts of chaos theory may of course be qualitatively useful in turbulence studies. One is the concept of "predictability". Qualitative arguments about the non-linear Navier–Stokes equations suggest that if two almost identical turbulence fields with the same boundary conditions are set up at time $t = 0$, then the two instantaneous velocity and pressure fields will become more and more different as time goes on, even though the statistical properties of the two fields will still be (nearly) equal. To a worker in turbulence, particularly an experimenter, this does not seem odd – but the issue of instantaneous versus statistical predictability has attracted a lot of attention in chaos studies, and perhaps our intuition about the Navier–Stokes equations may be put on a firmer footing. Deissler (1989) reviews applications of chaos studies in fluid dynamics; for a popular introduction to chaos studies in general, see the book by Gleick (1988); and see also, of course, the new interdisciplinary journal "Chaos".

11 Conclusions

In this paper, we have gone all the way from very basic questions of turbulence theory to the important practical question of the reliability of turbulence models, and then ended in chaos. The fallacies that we have discussed do not necessarily form a coherant story, but I think it can be said that most of them fall into the general category of unrigorous wishful thinking – the hope of finding simple solutions to a difficult problem. As H.L. Mencken said, "there is always an easy solution to every human problem – neat, plausible, and wrong". I never heard Keith Stewartson quote this, but it would certainly have appealed to him.

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TURBULENCE IN ASTROPHYSICS: Stars[1]

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ABSTRACT

Turbulence is ubiquitous in astrophysics, ranging from cosmology, interstellar medium to stars, supernovae, accretion disks, etc. Large scales and small viscosities combine to form large Reynolds numbers. Because it is not possible in a single article to review all the above scenarios, we limit ourselves to stars, in which thermal instabilities give rise to turbulent convection as the dominant heat transport mechanism. (Accretion disks, where shear instabilities dominate the outward transport of angular momentum, will be the subject of a second article, planned for Volume 31.) Because of the lack of a satisfactory theory, turbulence constitutes a bottleneck that prevents astrophysical models from being fully predictive. Because continued use of phenomenological turbulence expressions would make astrophysical models perennially unpredictive, a way must be found to make astrophysical models as prognostic as possible. In addition to the difficulties brought about by turbulence, astrophysical settings introduce "malicious conditions," of which the most refractory to a satisfactory quantification are compressibility (caused by the large density excursions that characterize convective zones in stars) and rotation. Basic understanding of how they affect turbulence in general is still rather sketchy. Reasons for the choice of stars and accretion disks as prototype examples are the following: The underlying instabilities are very basic; laboratory and direct numerical simulations data help constrain theoretical models; and new observational data, especially from helioseismology, help discriminate among different models with unprecedented accuracy.

1. TURBULENCE IN STARS

1.1 *General Features*

In the Universe, some 10^{23} ($\sim$Avogadro's number) stars exist, most of which, in different measure, rely on turbulence to transport heat when radiative processes are insufficient. One can roughly distinguish two classes of stars ($M_\odot$ = solar mass): Those of $M \leq 1M_\odot$ have a radiative core and a convective envelope, whereas stars with $M > 1M_\odot$ have a convective core and a radiative envelope. The Sun has a convective outer envelope roughly 30% of its radius with a mass $\sim 0.02M_\odot$; in a 10-$M_\odot$ star, the convective core is $\sim$25% of the radius and comprises $\sim$25% of the mass. Convection sets in depending on the entropy $\Sigma(r)$ gradient

$$\frac{T}{c_p}\frac{\partial\Sigma}{\partial r} = \frac{\partial T}{\partial r} - \frac{1}{c_p\rho}\frac{\partial p}{\partial r} \equiv \frac{\partial T}{\partial r} - \left(\frac{\partial T}{\partial r}\right)_{\rm ad} \equiv -\beta(r)\ . \qquad (1)$$

When $\partial\Sigma/\partial r > 0$ (< 0), convection is absent (present). The superadiabatic gradient is written as $\beta(r) = TH_p^{-1}(\nabla - \nabla_{\rm ad})$, where ∇ and $\nabla_{\rm ad}$ are dimensionless gradients, and $H_p = p/g\rho$ is the pressure scale height (at the base of the solar convective zone, $H_p \sim 10^9$ cm). Stellar interiors have large radiative conductivities χ (in the Sun, $\sim 10^9$ cm^2 s^{-1}) and a low kinematic viscosity ν. This entails a very low Prandtl number $Pr = \nu/\chi$ (in the Sun, $\sim 10^{-10}$). Thus, the Grashof number Gr easily exceeds $\sim 10^4$, above which convection becomes turbulent. Here, $Gr = Pr^{-1}Ra$, where $Ra = \lambda\beta\ell^4(\nu\chi)^{-1}$ is the Rayleigh number, $\lambda = g\alpha_v$, g is the local gravity, α_v is the volume expansion coefficient, and ℓ is a characteristic length. In laboratory convection, one maintains a fixed ΔT between the plates and measures the heat transported by convection. In stars, there are no rigid lids; the total flux $F_{\rm tot}$ originating from the star's interior is transported outward by radiative F_r and convective fluxes F_c, $F_r(r) + F_c(r) = F_{\rm tot}$. Once a model for F_c is chosen, the conservation equation yields the temperature profile $T(r)$ and hence the sound speed, which can be tested against helioseismological data. The challenge is to find an appropriate expression for F_c.

To set the stage for the discussion to follow, we recall that in convection there are two competing processes: Buoyancy accelerates blobs of hot gas on a time scale $t_b^{-1} = (\lambda\beta)^{1/2}$, while radiative processes try to deprive them of their heat content; the resulting cooling, on a time scale $t_\chi = \ell^2\chi^{-1}$, and the increased density make them prone to descending motion. The relevant parameter is the ratio $(t_\chi/t_b)^2 = Pr\,Ra \equiv S$; in the astrophysical literature one employs a convective efficiency Γ, $2\Gamma = (1+\frac{2}{81}S)^{1/2} - 1$. To construct F_c, we write $F_c = \chi_t\beta \equiv \chi\beta\Phi$, where χ_t is a turbulent conductivity and $\Phi \equiv \chi_t/\chi$ is a dimensionless measure of convective over radiative transport. To construct $\Phi(S)$, we begin with a local model whereby production equals dissipation, $P = \epsilon$, where $P = \lambda\overline{w\theta}$, and $F_c = \overline{w\theta}$ (w and θ are the fluctuating **u** and T fields). Because the rate of kinetic energy dissipation $\epsilon \sim u^2\tau^{-1} \sim \ell^2\tau^{-3}$, we derive $\Phi = (\lambda\beta\chi)^{-1}\ell^2\tau^{-3}$. When convection is efficient, blobs of fluids travel the distance ℓ without significant heat losses by radiative processes, $t_b < t_\chi$. The relevant time scale is t_b. Thus, $\Phi \sim S^{1/2}$, which leads to a convective flux F_c independent of χ. When convection is inefficient $t_b > t_\chi$, the shortest time scale is t_χ, and the appropriate choice for τ is now a reduced t_b: that is, the rate τ^{-1} is reduced by the ratio t_χ/t_b, $\tau^{-1} \sim t_b^{-1}(t_\chi/t_b)$, which in turn gives $\Phi \sim S^2$. In conclusion, one expects that any model will yield a function $\Phi(S)$ with the above asymptotic behavior.

1.2 *Modeling Turbulent Convection: Local Models, Mixing-Length Theory*

To compute the complete function $\Phi(S)$, one needs to know the eddy spectrum $E(k)$, which is one of the main goals of any turbulence theory. The mixing-length theory assumes that the whole turbulent spectrum can be approximated by a single large eddy of size $\ell \sim k_0^{-1}$,

$$E(k) \sim \delta(k - k_0)\ . \tag{2}$$

The resulting Φ is well known (Gough & Weiss 1976, Canuto 1996). Because the ratio between the largest eddies L and the first eddy to be affected by molecular dissipative processes, $\ell_d = (\nu^3\epsilon^{-1})^{1/4}$, is given by $L/\ell_d \sim Re^{3/4}$ and because in stars $Re > 10^{10}$, L/ℓ_d is so large as to make the mixing-length theory (MLT) one-eddy model a poor approximation. In addition, the mixing length $\ell = \alpha H_p$ introduces a free parameter. If one checks the MLT against laboratory data for high Ra turbulent convection (Castaing et al 1989), one finds fluxes that are off by 60–100% in the region where it is most crucial to have a correct convection theory (Canuto 1996).

1.3 *Modeling Turbulent Convection: Canuto & Mazzitelli Model*

To avoid Equation 2 and compute the full eddy spectrum $E(k)$, Canuto & Mazzitelli (1991) employed the EDQNM (Eddy Damped Quasi-Normal Markovian) closure model, which has been amply tested (Lesieur 1990). The three basic results are (*a*) the MLT underestimates the F_c in the region of high efficiency, (*b*) the MLT overestimates the F_c in the region of low efficiency, and (*c*) owing to the lack of a length scale in an incompressible flow, the only physical length scale is z, thus, $\ell = z$. Conclusions (*a*) and (*b*) imply that accounting for all the eddies causes a rotation of the MLT $\Phi(S)$: Because a change in the MLT α can cause only a translation of Φ but not a rotation, points (*a*) and (*b*) cannot be accommodated within the MLT. Canuto & Mazzitelli's predictions were tested against laboratory, numerical, and astrophysical data. In the first case, the model performed considerably better than the MLT. In the second case, validation comes from large-eddy simulation (LES) work (Demarque et al 1997), whereas for the astrophysical case, we have the following: red giants (Stothers & Chin 1995), helioseismology (Basu & Antia 1994, Baturin & Mironova 1995, Antia & Basu 1997), white dwarfs (Althaus & Benvenuto 1996), low-mass stars (D'Antona & Mazzitelli 1996), stellar atmospheres (Kupka 1996), and the $\ell \sim z$ rule (Stothers & Chin 1997). See more details in Section 2.

1.4 *Assessment of Two-Point Closures to Model Convection*

The Canuto & Mazzitelli (1991) model and the new version (Canuto et al 1996) account for the full eddy spectrum and for a degree of nonlocality via $\ell \sim z$. Diffusion, large-scale flows, rotation, and compressibility can be included, but the results are cumbersome. Previous authors also met with serious difficulties with inhomogeneous flows. It is fair to conclude that the traditional two-point closures have exhausted their fruitfulness. To make progress, one must resort to (*a*) numerical simulations, (*b*) one-point closures, and (*c*) new two-point closures.

1.5 *Numerical Simulations*

Because the astrophysical results have been reviewed recently (Spruit et al 1990), we only discuss the basic conceptual difficulties that remain unresolved. With the definitions given above, it is easy to derive the relations among the Reynolds, Prandtl, Rayleigh, and Péclét numbers ($Pe \sim w\ell\chi^{-1}$). They are

$$Pr\,Re^2 \sim Ra(\Phi S^{-1/2})^{2/3} \quad \text{and} \quad Pr\,Re \sim (S\Phi)^{1/3} \sim Pe\ . \tag{3}$$

Thus, the width of the eddy energy spectrum is given by

$$\frac{L}{\ell_d} \sim \left(\frac{Pe}{Pr}\right)^{3/4} \sim Pr^{-3/4}(S\Phi)^{1/4}\ . \tag{4}$$

In the case of efficient convection, $Pe > 1$, and with $Pr \sim 10^{-10}$ and $\Phi \sim S^{1/2}$, one has

$$\frac{L}{\ell_d} \sim 10^{10}\ , \quad N \sim \left(\frac{L}{\ell_d}\right)^3 \sim 10^{30}\ , \tag{5}$$

with a typical value $S \sim 10^8$. The large number of scales constitutes a severe limitation for LES because the number of grid points N is many orders of magnitude larger than the $N_{\max} \sim 10^{12}$ that the best computers can handle. Thus, LES resolves only a very minute fraction of the scales, and a huge number of them must be treated with a subgrid scale model (SGS), a requirement that is far from easy. Consider the well-known Smagorinsky-Lilly SGS model: The SGS are represented by a turbulent viscosity $\nu_t = (C_s \Delta)^2 S$, where S is the mean rate of shear and Δ is the smallest resolved scale. The model predicts a C_s twice as large as what is needed to explain channel flow data. Recently, a new model has been proposed that yields the correct C_s (Canuto & Cheng 1997). In stars, what is the correct value of C_s? The question was never answered; it may never have been asked. Because stellar turbulence is rather complex, the SGS model is today an essentially uncontrolled component whose effects on the final LES results are unknown. For a hierarchy of SGS models that tries to account for physical processes of astrophysical relevance, see Canuto (1994). It must also be stressed that an augmented kinematic viscosity to assure numerical stability is an even worse model, since Marcus (1986) has shown that it draws energy not from Δ but from the large scales L that ought to be immune from viscosity. Other models represent the SGS by a drag of the form $-\tau^{-1}\mathbf{u}$, and they assume $\tau^{-1} \sim Vk$, where V is the velocity of the largest scales. The SGS is interpreted as a kinematic rather than a dynamic effect (Canuto 1997a), an interchange that is known to have prevented the derivation of the Kolmogorov spectrum. Finally, because LES models are very time consuming, it is not possible to link them to a stellar structure code. LES can only be applied to a limited portion of a single star, like the convection zone (CZ) in the Sun, where it has found its most extensive and useful applications (Brummel et al 1995, Spruit 1997).

1.6 *One-Point Closure Models: Reynolds Stress Models*

GENERAL CONSIDERATIONS The one-point closure, also known as the Reynolds Stress Model (RSM), has a long tradition in fluid dynamics but has yet to be fully exploited in astrophysics. It has several advantages: versatility (it can be applied to any star), transparency of content (buoyancy, shear, stratification, rotation, etc, can be included), and ease of application (the equations can be solved with workstations). With the total velocity and temperature fields split as sums of mean $(T, \mathbf{U})$ and fluctuating parts $(\theta, \mathbf{u})$, one constructs the dynamic equations for the second-order moments $\overline{u_i u_j}$, $\overline{\theta^2}$, $\overline{u_i \theta}$, and ϵ. In the one-dimensional case, there are five independent turbulent variables:

$$K\ , \quad \overline{\theta^2}\ , \quad \frac{1}{2}\overline{w^2}\ , \quad \epsilon\ , \quad \overline{w\theta}\ , \tag{6}$$

i.e. turbulent kinetic energy, temperature variance (potential energy), kinetic energy in the z-direction (related to turbulent pressure, $p_{\text{turb}} = \rho\overline{w^2}$), dissipation rate, and convective flux. If A represents any of the second-order moments (Equation 6), the generic dynamic equation is of the form (Canuto 1992, 1993)

$$\frac{\mathrm{D}A}{\mathrm{D}t} + D_f(A) = \textit{sources} + \textit{distribution terms} - \textit{sinks}, \tag{7}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + U_j \partial/\partial x_j$. The diffusion D_f is defined as

$$D_f(A) \equiv \frac{\partial}{\partial x_k}(\overline{Au_k}) \ . \tag{8}$$

Because the As are second-order moments, the D_fs entail third-order moments (TOMs), for which one has essentially three choices: (*a*) $D_f = 0$, (*b*) down-gradient approximation (DGA), and (*c*) dynamic equations.

With (*a*), the model becomes local, whereas in (*b*) one writes, in formal analogy with the kinetic theory of gases,

$$\overline{Au_k} = -\nu_t \frac{\partial \overline{A}}{\partial x_k} \ , \tag{9}$$

where $\nu_t \sim w\ell$ is a turbulent diffusivity-viscosity and ℓ is a mixing length vaguely analogous to a mean free path. Equation 9 has been used in stellar convection (Xiong et al 1997), but it yields unphysical results in the buoyant planetary boundary layer. In (*c*), one avoids Equation 9 by deriving the dynamic equations for the TOMs. This entails fourth-order moments that are treated as in EDQNM (Canuto 1992). An analytical solution can be found for the stationary case (Canuto 1993). The main new physical aspect is that all the TOMs are the sum of terms like those of Equation 9 but with all the second-order moments rather than just one. For example,

$$\overline{w^3} = B_1 \frac{\partial}{\partial z}\overline{w\theta} + B_2 \frac{\partial}{\partial z}\overline{w^2} + B_3 \frac{\partial}{\partial z}\overline{\theta^2} + B_4 \frac{\partial}{\partial z}\overline{q^2} \ , \tag{10}$$

whereas in the DGA approximation (Equation 9), only the B_2 term survives; the diffusivities Bs are also given within the same model. The new TOMS were successfully tested against LES data (Canuto et al 1994b). Because stellar interiors exhibit large radiative conductivities, D_f must contain the radiative component, for example (the only other D^r is for $\overline{w\theta}$):

$$D_f^r(\overline{\theta^2}) = -\frac{\partial}{\partial z}\left(\chi \frac{\partial}{\partial z}\overline{\theta^2}\right) \ . \tag{11}$$

.

THE FIVE BASIC DYNAMIC EQUATIONS The five basic equations for Equation 6 are

$$\frac{\partial K}{\partial t} + D_f(K) = \lambda\overline{w\theta} - \epsilon \ , \tag{12}$$

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{\theta^2} + D_f\left(\frac{1}{2}\overline{\theta^2}\right) = \beta\overline{w\theta} - \epsilon_\theta , \tag{13}$$

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{w^2} + D_f\left(\frac{1}{2}\overline{w^2}\right) = c\lambda\overline{w\theta} - d\tau^{-1}\left(\overline{w^2} - \frac{2}{3}K\right) - \frac{1}{3}\epsilon , \tag{14}$$

$$\frac{\partial\epsilon}{\partial t} + D_f(\epsilon) = A\tau^{-1}\overline{w\theta} - B\epsilon\tau^{-1} , \tag{15}$$

and

$$\frac{\partial J}{\partial t} + D_f(J) = \beta\overline{w^2} + a\lambda\overline{\theta^2} - b\tau^{-1}J , \tag{16}$$

where $J \equiv \overline{w\theta}$, $\tau \equiv 2K\epsilon^{-1}$, and $A, B, a, c,$ and d are constants (Canuto 1992).

In the absence of large-scale flows, there are only two sources, $\lambda\overline{w\theta}$ and $\beta\overline{w^2}$; ϵ_θ plays the role of ϵ. Equations 12–16 must be solved together with the dynamic equation for the mean temperature

$$\frac{\partial}{\partial t}(c_p T + K) = -\frac{\partial}{\partial x_i}\left(F_i^r + F_i^c + F_i^{\kappa e}\right) + Q_{\text{ext}}, \tag{17}$$

where Q_{ext} represents any external energy sources. The problem is now fully specified: All five turbulent variables (Equation 6) and the mean T are described by differential equations. The diffusion terms are given in Canuto (1992, 1993). In the local limit, the above five equations can be solved analytically, and the resulting F_c is pretty close to the Canuto & Mazzitelli model (Canuto 1996). The strategy for solving the equations suggests itself. At $t = 0$, one can use the results obtained (for a given star) using a local model. The behavior in the overshooting (OV) region (see below) can be exponential decay laws for all variables (Equation 6), with a negative J. Evolving the equations in time, one should reach a stationary solution. It may not be advisable to assume $\partial/\partial t = 0$ and solve a boundary value problem, for the many nonlinear terms may lead to instabilities—not to mention that in the initial value problem, one does not have to guess the OV region and iterate on it. Equations 12–17 are presently being used to study undershooting in the solar CZ.

1.7 *Previous Nonlocal Models*

All previous nonlocal models (for a review, see Canuto 1993) employ one or two variables instead of the five in Equation 6, and the corresponding dynamic equations are phenomenological. Xiong et al (1997) use four turbulence variables, but ϵ is treated locally, thus introducing a free length scale; the TOMs were treated with DGA. These past models can be derived as simplified cases of Equations 12–16 (Canuto 1993). We discuss two models (Gough 1976, Roxburgh 1989) because they are still in use. The first model contains a differential equation (DE) for the variable $\overline{w\theta}$. The suggested equation is

$$\frac{\partial^2}{\partial \xi^2} F_c = p_1\left(F_c - F_c{}^{MLT}\right), \tag{18}$$

where $\xi = z/\ell$, $\ell = \alpha H_p$, and p_1 is constant. Using the following approximations:

$$\overline{w\theta^2} \rightarrow DGA \rightarrow -\nu_t \frac{\partial}{\partial z} \overline{w\theta} \ , \tag{19}$$

$$\nu_t^{-1}\left(\beta \overline{w^2} + \frac{2}{3}\lambda \overline{\theta^2}\right) \rightarrow F_c^{MLT} \ , \tag{20}$$

Equation 16 can be transformed to

$$\frac{\partial^2}{\partial \xi^2} F_c + \ell q \frac{\partial}{\partial \xi} F_c = p_1\left(F_c - F_c{}^{MLT}\right), \tag{21}$$

$$q \equiv \frac{1}{\nu_t} \frac{\partial}{\partial \xi} \nu_t \ . \tag{22}$$

A consistency problem arises: For Equation 20 to hold, $\nu_t \neq$ constant (specifically, it must be $\sim \ell K^{1/2}$), but for the second term in Equation 21 to vanish, $\nu_t =$ constant. Gough's model has been used by Balmforth (1992a) and Houdek (1997) to study solar pulsations. The second model was suggested specifically for overshooting, which we now discuss.

1.8 *Overshooting (OV)*

Within most of the convective zone, turbulence is so efficient that turbulent diffusion is larger than radiative diffusion, and thus D_f^r can be neglected. The final result is independent of χ and corresponds to the $\Phi \sim S^{1/2}$ limit discussed earlier. In a pragmatic sense, when turbulence is so efficient, a model of convection becomes unnecessary, for ∇ is nearly adiabatic, as detailed models indeed show: $\nabla = \nabla_{\text{ad}} + O(10^{-8})$. On the other hand, when buoyancy forces vanish, $\nabla = \nabla_{\text{ad}} + O(1)$, one encounters a new phenomenon, overshooting, the extent traveled inside the stably stratified region by the eddies propelled by their nonzero velocities. The only reliable constraints are from observations: Stothers & Chin (1991) used Population I stars to conclude that OV $< 0.2H_p$, whereas Basu (1997), using helioseismological data, concludes that OV $< 0.05H_p$. Both are upper limits. For more on these constraints, see Section 2.1. From a theoretical point of view, the quantification of the OV has been a contentious issue (Baker & Kuhfuß 1987, Roxburgh 1989) and the problem remains essentially unresolved. The reason is quite apparent: Five DE must be solved for the five variables (Equation 6), and this has not yet been done. The main physical features to be accounted for are as follows.

DISSIPATION Because of its braking action, dissipation is clearly a basic process. Two models were used in the past: $\epsilon = 0$, which is unphysical, and $\epsilon = \frac{1}{2}C_d w^3 \ell^{-1}$, which faces three problems. It is local, rather than nonlocal; the determination of the "drag coefficient" C_d is difficult (it decreases $\sim Re^{-1}$ up to $Re \sim 10^5$, but then saturates; Frisch 1995), and ℓ introduces an unknown coefficient, $\ell = cH_p$. Such local expression ought to be forsaken in favor of Equation 15.

DIFFUSION In the OV region, stratification is stable $\beta < 0$, and $\overline{w\theta} < 0$: The convective flux is thus a sink of K, Equation 12, but a source of potential energy, Equation 13. In the OV region, kinetic energy (KE) is transformed into potential energy (PE). The right-hand side of Equation 12 is negative; that is, there is no local source of kinetic energy. To obtain a stationary solution, the only source is the diffusion term, which brings energy from the unstable region, a nonlocal process. This calls for a reliable quantification of the TOMs.

POTENTIAL ENERGY Because in the OV, KE $\to$ PE, the latter cannot be neglected. This implies that the temperature gradient cannot be adiabatic. In fact, $\overline{w^2\theta}$ that enters in $D_f(J)$, Equation 16, vanishes at the beginning and the end of the OV. Thus, there is a point in the OV region where its derivative is zero, $D_f = 0$. Equation 16 with $\beta = -|\beta|$ and $\overline{w^2}\tau \sim \nu_t \sim \chi_t$ then gives

$$bJ = -\chi_t|\beta| + a\lambda\tau\overline{\theta^2} \ . \tag{23}$$

Because the last term is positive and to assure that $J < 0$, β must be nonadiabatic.

Consider now Equation 12 in the stationary case. Using the flux conservation law extended to include the flux of turbulent kinetic energy $F^{ke} = \frac{1}{2}\overline{wq^2}$, $q^2 = u_i u_i$, one can obtain a DE for F^{ke} that can be integrated. Because $F^{ke}(r_1, r_2) = 0$ (beginning and end of the CZ), one derives the following result (we introduce spherical coordinates and define the luminosities $L = 4\pi r^2 F$; the subscript N stands for the total, nuclear flux):

$$\int_{r_1}^{r_2} T^{-2} \left| \left(\frac{\partial T}{\partial r}\right)_{\mathrm{ad}} \right| (L_N - L_r)\Psi(r)\mathrm{d}r = \int_{r_1}^{r_2} 4\pi r^2 T^{-1} \rho\epsilon(r)\Psi(r)\mathrm{d}r \ , \tag{24}$$

where

$$\ln \Psi(r) = -\int T^{-1}\beta(r)\mathrm{d}r \ . \tag{25}$$

The full interval r_1–r_2 comprises the region r_1 to r^*, where $L_N - L_r > 0$, $\beta(r) > 0$, $\Psi \equiv \Psi_1 < 1$, and the OV region, r^* to r_2, where $L_N - L_r < 0$, $\beta <$

$0,\ \Psi \equiv \Psi_2 > 1$. Thus Equation 24 becomes

$$\int_{r_1}^{r^*} T^{-2}\left|\left(\frac{\partial T}{\partial r}\right)_{\rm ad}\right| |L_N - L_r|\Psi_1(r)\mathrm{d}r$$
$$= \int_{r^*}^{r_2} T^{-2}\left|\left(\frac{\partial T}{\partial r}\right)_{\rm ad}\right| |L_N - L_r|\Psi_2(r)\mathrm{d}r + \textit{Diss. Term},$$

$$\textit{Diss. Term} \equiv \int_{r_1}^{r_2} 4\pi r^2 T^{-1}\rho\epsilon(r)\Psi(r)\mathrm{d}r > 0\ . \tag{26}$$

Several conclusions are in order: (*a*) because ϵ is always positive, the OV extent $r_2 - r^*$ is less than without ϵ; (*b*) because $\Psi_2 > \Psi_1$, nonadiabaticity contributes more to the integral on the right-hand side than to the left-hand side. Thus, again, the OV extent is less than if the two Ψs were the same. Roxburgh (1989; see also Baker & Kuhfuß 1987) assumed $\epsilon = 0$ and $\Psi = 1$, neither of which can be justified and both of which lead to overestimate the OV. In summary, Equation 26 is exact and helps elucidate what is required. However, unless one has a model for $\epsilon = \epsilon(r)$ and $\beta(r)$, Equation 26 cannot be used to compute the OV.

1.9 *Differential Rotation and Convection: A Test of the RSM*

The Sun rotates differentially, spinning faster at the equator than at the poles. The long-term average surface Doppler rotation rate from Mount Wilson measurements is given by $\Omega_s(\theta) = \Omega_0(1 - a\cos^2\theta - b\cos^4\theta)$, where the latitude is $\pi/2 - \theta,\ \Omega_0 = 2.87\,\mu\text{rad s}^{-1},\ a = 0.12,\ b = 0.17$. Helioseismic inversion has shown that the variation with latitude in $\Omega(r,\theta)$ is approximately the same throughout the CZ (cf Section 2.6). Because $U_\phi = r\sin\theta\,\Omega(r,\theta)$, the above law ought to be derived from the dynamic equation governing the angular momentum, $L_z = U_i\eta_i,\ \eta_i = -y,\ x,\ 0$,

$$\frac{\mathrm{D}}{\mathrm{D}t}L_z = -g_i\eta_i + \frac{\partial}{\partial x_j}[\nu\eta_i U_{i,j} - \eta_i(\tau_{ij} + P\delta_{ij})]\ . \tag{27}$$

Because ν is small, the main contribution comes from the Reynolds stress $\tau_{ij} \equiv \overline{u_i u_j}$, which can then be directly tested. Alternatively, one can assume the observed $\Omega(r,\theta)$, compute τ_{ij}, and compare the results with sunspot data (Gilman & Howard 1984, Pulkinnen et al 1993).

The underlying physical question is as follows: Can turbulent convection that exists owing to thermal instabilities generate a mean flow U_ϕ of the observed form? (In standard laboratory situations, it is shear that generates turbulence.) In terms of energy, can part of the buoyancy production P_b ($\equiv \lambda\overline{w\theta}$) be used to generate U_ϕ rather than being all dissipated into heat by molecular processes? Can we write $P_b + P_s = \epsilon$, with a negative shear production

$P_s = -2\overline{u_\theta u_\phi} S_{\theta\phi}$, $S_{\theta\phi} \equiv \sin\theta\, \partial\Omega/\partial\theta$, that acts like a sink? In the northern hemisphere, Ω increases toward the equator, $\sin\theta\, \partial\Omega/\partial\theta > 0$; because we require $P_s < 0$, we must have $\overline{u_\phi u_\theta} > 0$. Thus, models of the type

$$\overline{u_\theta u_\phi} = -\nu_t \sin\theta \frac{\partial\Omega}{\partial\theta} \tag{28}$$

are not acceptable. To remedy the situation, the so-called Λ-effect was suggested (Rüdiger 1989), whereby one empirically extends Equation 28 to

$$\overline{u_\theta u_\phi} = \Lambda \cos\theta - \nu_t \sin\theta \frac{\partial\Omega}{\partial\theta}\ , \tag{29}$$

which can be made to fit the data. To make the model more prognostic, Durney (1996) has adapted the MLT model to this highly anisotropic situation.

The full RSM needs no tweaking, for it is supposed to contain the basic physics: It is therefore a critical issue to test the model with the data. Because the RSM includes vorticity, $R_{\theta\phi} = -S_{\theta\phi} - 2\Omega\cos\theta$, which depends on Ω itself, the Λ-effect is automatically accounted for. Thus, it is to be expected that the interaction between buoyancy and vorticity will play a key role. This is indeed the case. Solution of the full RSM (Canuto et al 1994a) shows (cf Figure 1) that with shear only (model 1) and buoyancy + shear (model 2), the $\tau_{\theta\phi}$ have

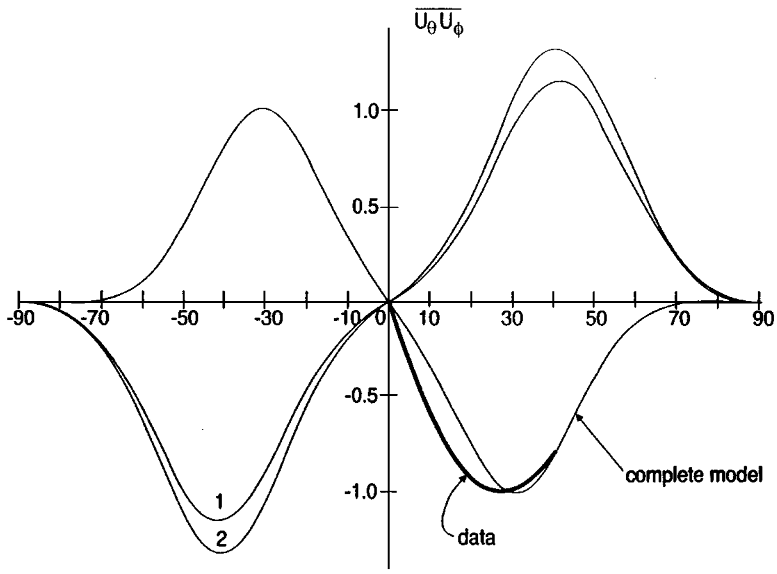

Figure 1 The Reynolds stress $\overline{u_\theta u_\phi}$ vs colatitude. Models 1 and 2 that do not include vorticity yield an incorrect result. When the buoyancy-vorticity interaction is introduced, the computed $\overline{u_\theta u_\phi}$ has the right sign and magnitude.

the wrong sign; when vorticity is included, the signs automatically switch and agreement with the data is obtained. It is important to emphasize that the latter was not obtained by adjusting any parameters, since the values employed were the ones determined by previous applications of the RSM to a large variety of different flows. The model also predicts all the other Reynolds stresses, but the lack of data prevents a direct comparison. Heat fluxes $\overline{u_i\theta} = \chi_{ij}\beta_j$ are characterized by a tensorial conductivity tensor χ_{ij} for which we have the following results: χ_{rr} and $\chi_{\phi r}$ are symmetric, whereas $\chi_{\theta r}$ is antisymmetric; $(\chi_{\theta r}, \chi_{\phi r}) \sim 0.1\chi_{rr}$, which is positive for any θ; the maxima of $\chi_{\theta r}$ and $\chi_{\phi r}$ occur at 45° and 35°, respectively; $\chi_{\phi r} < 0$ always; $\chi_{\phi r}$ and $\chi_{\theta r}$ vanish at the poles and $\chi_{\theta r}$ at the equator; the convective flux $\overline{u_\theta\theta}$ is directed away from the equator toward the poles; $\overline{u_\phi\theta} < 0$ everywhere. Finally, the model shows that the maximum buoyancy production is at 40°, which may be taken as the location where differential rotation originates (the shear $S_{\theta\phi}$ is maximum at 50–60°). The model also predicts that about 2.5% of the buoyancy goes into generating differential motion.

In conclusion, from both a physical point of view as from the purely pragmatic aspect of fitting the data, the RSM was shown to contain the necessary ingredients. In the next step, one should reverse the problem and try to deduce the function $\Omega(r, \theta)$ by solving the angular momentum equation together with the RSM equations.

1.10 *Rigid Rotation and Convection*

Coriolis force, by making the wave vectors **k** rotate around different axes, weakens their interactions and thus the efficiency of the nonlinear interactions, which transfer energy from the largest to the smallest eddies. The spectrum is in fact $E(k) \sim (\epsilon\Omega)^{1/2}k^{-2}$ up to $k_\Omega \sim \ell_\Omega{}^{-1} = (\Omega^3\epsilon^{-1})^{1/2}$, after which the Kolmogorov spectrum $E(k) \sim \epsilon^{2/3}k^{-5/3}$ applies. At the base of the solar CZ, $\epsilon \sim 1\ \mathrm{cm}^{-2}\ \mathrm{s}^{-3}$, and $\ell_\Omega \sim 10^9$ cm $\approx H_p$. On these general grounds, the convective flux is reduced by rotation. Except for early work (Canuto & Hartke 1986), we are not aware of any model that has provided the required flux $F_c = F_c(S, \Omega; Pr \ll 1)$. Both experimental (Rossby 1969) and theoretical work (Brummel et al 1996, Tilgner & Busse 1997) deals with *Pr* too large for stars. The stochastic model (Canuto & Dubovikov 1996) is presently being extended to include rotation.

1.11 *Compressibility*

Stellar interiors are compressible, for the density varies greatly within the CZ. Accounting for compressibility only via the Boussinesq approximation has long been lamented, but no model has yet been presented that does any better, except for direct numerical simulations (Chan & Sofia 1986, Cattaneo et al

1991, Hossain & Mullan 1991). Attempts to use a two-point closure model (Hartke et al 1988, Schilling 1994) showed that the problem can rapidly become unmanageable but also that the longitudinal kinetic energy is a sizable fraction of the incompressible one and thus hardly negligible. Since then, RSMs were used (Yoshizawa 1995, Xiong et al 1997), but they employed the DGA to treat TOMs, and ϵ was taken to be the incompressible value. From the physical viewpoint, one expects that a div $\mathbf{u} \neq 0$ will increase dissipation; that is, the true dissipation is $\epsilon = \epsilon_s + \epsilon_c$, where the solenoidal (incompressible) component is now increased by a dilatation (compressible) part $\epsilon_c \sim M^2\epsilon_s$, where M is the Mach number. This effect will cause a decrease of the OV. Recently, a RSM was worked out in which full compressibility, time-dependence, nonlocality, and a large-scale flow were included. The dynamical equations for all the TOMs were also derived. The closure was performed at the fourth-order moments (Canuto 1997b). The model awaits numerical solution.

1.12 *New Turbulence Models*

No RSM has ever been derived from a two-point closure, and yet such solution would be highly desirable because it would fix the few adjustable parameters that present RSMs cannot determine and would unify two procedures. Progress has been made with a new two-point closure (Canuto & Dubovikov 1996, 1997). The method begins with the case of homogeneous, isotropic turbulence and employs renormalization group (RNG) techniques to sum the infinite set of Wyld-Dyson (WD) diagrams to represent the ultraviolet part of the spectrum. The infrared part is dealt with differently. The resulting model, which has no free parameters, is then extended to anisotropic and inhomogeneous flows and tested against a large variety of data. It explains more than $\sim$70 statistics, including shear, buoyancy, two-dimensional turbulence, rotation, etc. Recently, the model was integrated over all wave numbers. Three important results ensued: The dynamic equations exhibit the RSM form; the time scales that the RSM is unable to fix are fully determined, as are the few adjustable parameters. The dependence on the degree of convective efficiency *Pe* is crucial to yield a credible OV.

2. SEISMIC TESTS OF STELLAR TURBULENCE

2.1 *Introduction*

The most obvious effects of turbulence in stellar structure and evolution are associated with convection, as discussed in Section 1. In computations of stellar models, an expression for the convective flux is required, as a function of the temperature gradient; additional features, which are often ignored, include the dynamical effects on the structure and overshoot and other transport processes beyond the convectively unstable regions. The currently used models of the convective flux often contain undetermined parameters, such as the mixing length (cf Section 1.2). This parameter only affects the near-surface region where the stratification deviates noticeably from being adiabatic; adjusting the parameter has the effect of changing the adiabat of the essentially adiabatic bulk of the CZ. In the solar case, this adjustment is used to ensure that the model of the present Sun has the correct radius; the resulting models are then largely independent of the details of the convection treatment (e.g. Gough & Weiss 1976). Even the Canuto & Mazzitelli (1991) formulation, where in principle the relevant length scale is determined, requires a slight adjustment to obtain solar models with a radius consistent with the measured value. The length parameter obtained in the solar case is then often used for other stars. Whether or not this essentially arbitrary procedure is in accordance with observations is somewhat unclear. Several calculations of evolved models of relatively low-mass stars (e.g. VandenBerg 1992, 1997; for a review, see also VandenBerg et al 1996) have shown reasonable agreement with observations of the red-giant branches (RGB) of open clusters, using the solar calibration of MLT. On the other hand, for somewhat more massive stars, Stothers & Chin (1995) found that substantial variation of the parameter of MLT was required to match the RGB, whereas the Canuto & Mazzitelli formulation provided a good match with no adjustment.

The LES models of the top of the solar CZ essentially cover the superadiabatic part of the CZ and hence provide a prediction of the adiabat in the adiabatic part of the CZ. Interestingly, the value is rather close to that obtained by calibrating simple models (e.g. Christensen-Dalsgaard 1997). This suggests the possibility of determining effective values of the parameters of the simple theories, as functions of stellar parameters, through matching to results of LES (e.g. Ludwig et al 1997, Trampedach et al 1997).

Dynamical effects, often described as a turbulent pressure $p_{\rm turb}$ (cf Equation 6), may contribute a substantial fraction of the total hydrostatic pressure in near-surface regions of CZs, where the motion approaches sonic speeds. This might considerably affect the structure of the top of the CZ. Overshoot beyond the unstable region causes changes in the thermal structure; however, for stellar evolution, a more important effect is the mixing caused by OV from convective cores. With the present formulations, the OV distance is usually taken as a fraction, $\mathrm{OV} = \alpha_{\rm ov} H_p$, of the pressure scale height H_p at the edge of the core. Comparison with observations of models without and with core OV has given somewhat ambiguous results: Andersen et al (1990) found strong evidence for OV (with $\alpha_{\rm ov} = 0.25$) from analysis of well-observed binary stars; however, Stothers & Chin (1991) pointed out that with updated opacities, there was no clear evidence for OV at a level of $0.2H_p$. Very recently, Nordström et al (1997) carried out a careful analysis of the color-magnitude diagram of an open cluster, comparing their results with models with up-to-date opacities; they again found a strong preference for models with $\alpha_{ov} \simeq 0.2$.

Although convection is the most obvious source of stellar turbulence, it is not the only one. Observational evidence strongly indicates that the rotation rate of solar-like stars decreases with age. Models of the history of solar rotation therefore generally assume that the Sun started its evolution in a state of rapid rotation; angular momentum was lost through the solar wind, with the main coupling to the CZ. Because helioseismic evidence (cf Section 2.6) shows that the solar interior is rotating almost uniformly at close to the present surface rate, some mechanisms must transport angular momentum in the solar interior. This may well involve instabilities that lead to turbulence and hence mixing, which might have a significant effect on solar evolution.

"Classical" observations of stellar properties such as radius, luminosity, mass, and age provide only fairly crude tests of stellar modeling and of the influence of turbulence. Much more extensive data are required if we are to test the details of turbulence discussed in Section 1. Such data are provided by the dynamical properties of stars, as reflected by motion in their atmospheres and by large-scale pulsations. Because of its proximity, the most detailed observations by far are available for the Sun.

High-resolution measurements of the solar surface structure and velocity fields immediately show motion and intensity fluctuations associated with convection, in the granulation and supergranulation. (It is interesting that the results of detailed LES of the upper part of the CZ show a striking similarity with the properties of granulation; e.g. Stein & Nordlund 1989.) However, these data contain much more information of a more subtle nature. Careful analysis has shown the presence of a very large number of highly regular oscillations of the solar surface, akin to the motions set off in the Earth by large earthquakes. As in the terrestrial case, the frequencies of these oscillations carry information about the structure and other properties of the solar interior. However, unlike those of the Earth, the oscillations of the Sun are maintained continuously; they are believed to be excited by the nearly sonic turbulence of the convection just beneath the solar surface. And, as we cannot on the Earth, we can measure the oscillation everywhere on the visible hemisphere of the Sun; for example, the Solar Oscillations Investigation/Michelson Doppler Imager (SOI/MDI) project on the Solar and Heliospheric Observatory (SOHO) satellite (e.g. Kosovichev et al 1997) is carrying out simultaneous observations at about 700,000 locations on the Sun, far exceeding the possibilities for measurement of oscillations of the Earth.

The oscillations are observed by means of several techniques, of which the most detailed observations rely on Doppler velocity. Periods are approximately 3–10 min, and the spatial scales range from radial modes, where the Sun expands and contracts spherically, to modes with a surface wavelength of $\sim 10^3$ km. The velocity amplitudes are minute, less than about 15 cm s^{-1} for the individual modes, which corresponds to luminosity oscillations (which have also been measured) on the order of a few parts per million.

Physically, the oscillations correspond to standing acoustic waves and in addition, at shorter spatial scales, to surface gravity waves (f modes). The frequencies can be determined with extreme accuracy, in many cases exceeding 10^{-6}. Inverse analyses have allowed the determination of the sound speed and density of the solar interior, thus providing detailed tests of solar models; in addition, the internal rotation has been measured in considerable detail, leading to the rejection of earlier calculations of the redistribution of angular momentum within the solar CZ. Finally, amplitudes and lifetimes of the modes carry information about the excitation processes and hence about the properties of the vigorous near-surface convection.

In the following, we present some aspects of such helioseismic investigations and discuss the relevance of the results for the modeling of turbulence in the Sun. For more detailed presentations on helioseismology, see, for example, Gough & Toomre (1991) and Gough (1993).

2.2 *Basic Aspects of Helioseismology*

The normal modes of the Sun depend on colatitude θ and longitude ϕ as spherical harmonics $Y_l^m(\theta, \phi)$, characterized by the degree l and the azimuthal order m, such that $|m| \leq l$. Physically, l is related to the length $k_{\rm h}$ of the horizontal wave number at a distance r from the solar center by $k_{\rm h} = [l(l+1)]^{0.5}/r$, modes with $l = 0$ being spherically symmetric; m measures the number of nodes around the equator of the star. For each l and m, solar models allow a set of modes characterized by the radial order n, which measures the number of nodes in the radial direction, and with frequencies ω_{nlm}.

Had the Sun been strictly spherically symmetric, the frequencies would have been independent of m, for in that case the location of the equator on which m depends would have had no physical meaning. The frequencies ω_{nl} are then determined just by the spherically symmetric structure of the star. This degeneracy is lifted by departures from spherical symmetry; in particular, rotation induces a frequency perturbation

$$\Delta\omega_{nlm} = m \int_0^R \int_0^\pi K_{nlm}(r, \theta)\Omega(r, \theta) r \, \mathrm{d}r \, \mathrm{d}\theta \ , \qquad (30)$$

where $\Omega(r, \theta)$ is the angular velocity of the solar interior; the kernels K_{nlm} depend on the displacement field in the mode (they roughly correspond to the energy density) and may be calculated for a given solar model. Note that the extent in latitude of the Y_l^m, and hence of the kernels K_{nlm}, depends on m:

sectoral modes with $|m| = l$ are concentrated near the equator while zonal modes with $m = 0$ extend over all latitudes; hence, the variation with m of $\Delta\omega_{nlm}$ contains information about the latitude dependence of Ω.

As a result of their small amplitudes, the oscillations are described very accurately by linear theory as perturbations around an equilibrium stellar configuration. Furthermore, in almost the entire star, the oscillations can be regarded as isentropic, with high precision: The relevant thermal time scales are much longer than the typical pulsation period. Such oscillations can be calculated easily and with high precision for a given stellar model; as a result, the frequencies provide direct information about the solar interior.

The properties of the acoustic modes can be understood in terms of the behavior of plane acoustic waves, as they propagate through the varying sound speed c in the solar interior: Near the surface they propagate predominantly vertically; owing to the increasing sound speed, they are gradually refracted, reaching a point of total internal reflection at the so-called turning point, whose distance r_t from the solar center is determined by $c(r_t)/r_t = \omega/[l(l+1)]^{0.5}$. (Spherically symmetric waves, with $l = 0$, propagate through the solar interior with essentially no internal reflection.) At the surface, the waves are similarly reflected by the rapid decrease in density. The resulting standing waves form the normal modes of the Sun. Over the observed range of ω and l, the location of the lower turning point ranges over most of the solar interior. This variation of the part of the Sun to which the modes are sensitive makes possible the determination of the local properties of the solar interior from the observed frequencies. The frequencies of the f modes are largely determined by the solar surface gravity and k_h, with little dependence on the details of solar structure.

The physical description of the near-surface layers suffers from uncertainties in modeling of the structure and oscillations in this region (cf Section 2.5 below), which are intimately connected with the description of the vigorous turbulence in the upper part of the CZ. However, the effects on the frequencies generally have a simple dependence on the degree and frequency of the modes. In the superficial layers, the acoustic waves propagate almost vertically. As a result, the physical influence of the near-surface problems is essentially independent of the degree. The effects on the frequencies do depend on degree, however, because low-degree modes penetrate more deeply, involve a larger fraction of the Sun, and hence are more difficult to perturb than are high-degree modes. This essentially trivial dependence can be eliminated by scaling the frequency differences by $Q_{nl} = E_{nl}/\bar{E}_0(\omega_{nl})$, where E_{nl} is the inertia of the mode and $\bar{E}_0(\omega)$ is the inertia of a radial mode of frequency ω (e.g. Christensen-Dalsgaard & Berthomieu 1991); in this way, the effects are essentially renormalized to the corresponding effects on radial modes. At low frequencies, the upper reflection

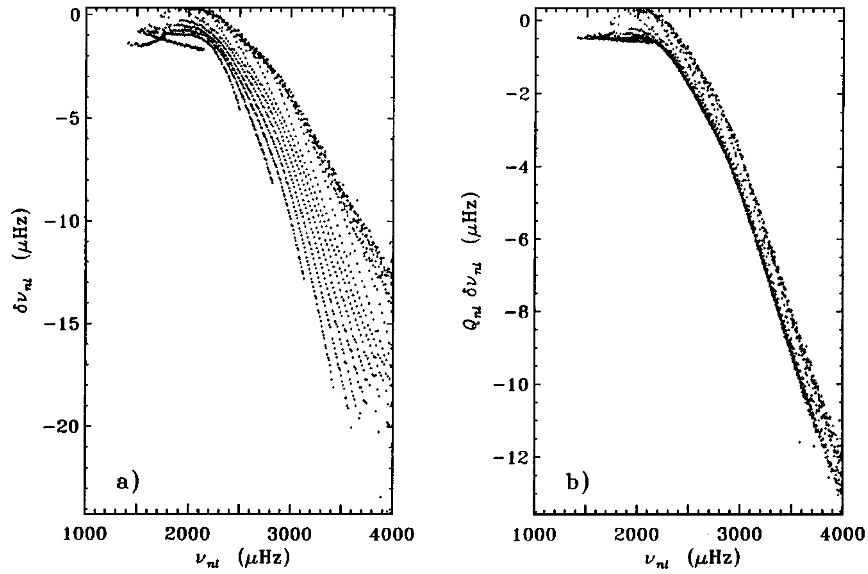

Figure 2 Differences between observed solar frequencies and frequencies of a solar model, in the sense (Sun)–(model), plotted against cyclic frequency ν. The observations were obtained with the Global Oscillation Network Group (GONG) network (Harvey et al 1996). The model is Model S of Christensen-Dalsgaard et al (1996). The computation included settling of helium and heavy elements, as well as up-to-date opacity and equation of state; convection was treated with MLT, neglecting turbulent pressure. Panel *a* shows unscaled differences, whereas in *b* the differences have been scaled by the inertia ratio Q_{nl}, to highlight the dominant effects of near-surface errors.

point of the modes is generally below the problematic region; hence, for such modes, the effect of the near-surface errors is expected to be small.

To illustrate the effects of the scaling, Figure 2 shows unscaled and scaled differences between solar frequencies and the frequencies of a typical solar model. The scaled differences are evidently in fact largely a function of frequency; they are small at low frequency, which indicates that the dominant differences between the Sun and the model are indeed associated with the near-surface layers.

Inferences about the solar interior from the observed frequencies involve inverse analyses, using techniques originally developed in geophysics. The procedures are most easily illustrated for the linear relation between the observed frequency splitting $\Delta\omega_{nlm}$ and the unknown internal angular velocity $\Omega(r, \theta)$ in Equation 30. A simple method is to make a least-squares fit to the data of a discretized representation of Ω. As different modes provide nearly the same information about the internal rotation, the problem is highly ill conditioned;

hence, the fit must be regularized, e.g. by also minimizing some measure of smoothness of the solution, such as the integral of the squared second derivatives of Ω (see, for example, Schou et al 1994 for details). Conceptually different techniques involve the explicit construction of linear combinations of the data in Equation 30, such that in the corresponding right-hand sides of that equation the combinations of the kernels K_{nlm} are localized near some target points (r_0, θ_0); in this way one obtains a localized average of Ω near the target (e.g. Pijpers & Thompson 1992). These so-called optimally localized averages (OLA) techniques must again be regularized, to control the magnitude of the error in the inferred solution.

Inversion for solar structure is generally done in a differential fashion: Frequency differences between the Sun and a model are expressed in terms of, for example, the differences $\delta c^2/c^2$ and $\delta\rho/\rho$. Again, through suitable combinations of the differences for the observed modes, obtaining localized measures of $\delta c^2/c^2$ is possible while suppressing the contributions to the differences from $\delta\rho/\rho$, from the near-surface errors illustrated in Figure 2 and from observational error.

2.3 *Structure of the Radiative Interior*

Figure 3 shows the difference in squared sound speed between the Sun and a solar model, inferred in this manner. A finite set of data can never give complete information about the variation of sound speed with position. This limitation is reflected by the fact that the results provide localized averages of $\delta c^2/c^2$, of an extent indicated by the horizontal bars. Although the differences are generally small, by the normal standards of astrophysics, they are nevertheless very substantial compared with the estimated errors. Thus, there are evidently highly significant problems with the standard solar model.

Unfortunately, there is no unique cause for these discrepancies. Much of the sound-speed difference can be accounted for in terms of changes of the opacity of a magnitude well within the generally assumed uncertainty in current opacity calculations. However, given the uncertain treatment of possible hydrodynamical phenomena in the solar radiative interior, such phenomena are likely to contribute. In particular, it is striking that a dominant feature in the sound-speed difference is located in a region of steep gradients in composition, caused by the gravitational settling of helium and heavier elements from the CZ. Weak mixing would decrease the gradient, hence increasing the hydrogen abundance and thereby the sound speed at the peak in $\delta c^2/c^2$. Such mixing might be caused by shear instabilities associated with the braking of the likely early rapid solar rotation (e.g. Chaboyer et al 1995), or by the processes causing the sharp gradient of angular velocity in the tachocline at the base of the CZ (cf Section 2.6) (Spiegel & Zahn 1992, Elliott 1997). Weakly turbulent

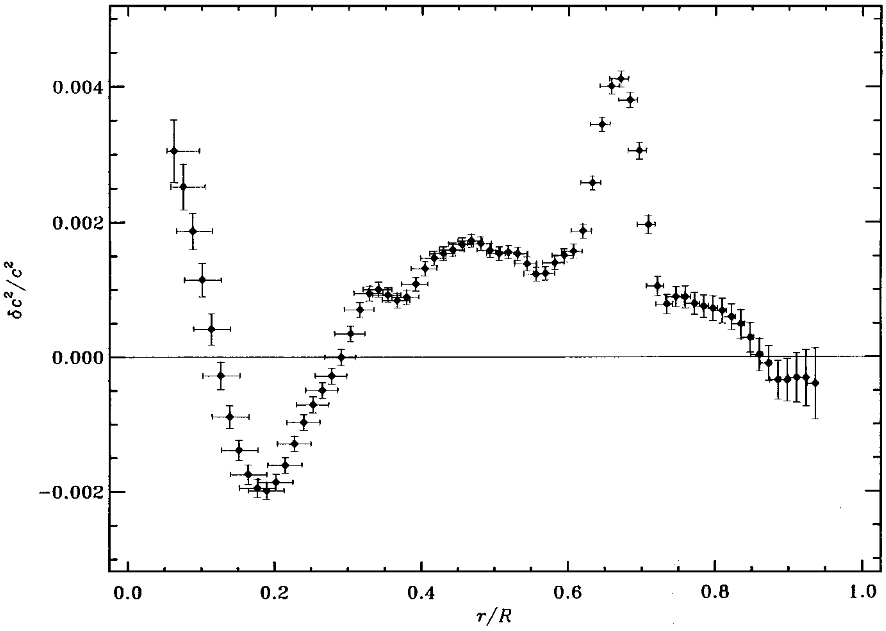

Figure 3 Difference in squared sound speed between the Sun and Model S (cf the caption to Figure 2), in the sense (Sun)–(model), obtained by inverting frequency differences between the Sun and the model. The observed frequencies were a combination of data from the Birmingham Solar Oscillation Network (BiSON) (Chaplin et al 1996) and the High Altitude Observatory's LOWL instrument (Tomczyk et al 1995a). The abscissa is distance from the solar center, in units of the surface radius. The vertical error bars show 1σ errors in $\delta c^2/c^2$, inferred from the errors in the observed frequencies, whereas the horizontal bars provide a measure of the resolution of the inversion. (From Basu et al, submitted for publication.)

mixing might also be caused by penetration from the CZ, possibly in the form of intermittent plumes penetrating to substantial distances. In fact, independent evidence for mixing well beyond the CZ is provided by the depletion of lithium and beryllium in the solar photosphere, relative to the initial abundance as reflected in meteorites (e.g. Anders & Grevesse 1989).

A second notable feature in the sound-speed difference is the negative value at the edge of the nuclear-burning core, around $r \simeq 0.2R$, followed by an apparent rise closer to the center. This may again be a sign of partial mixing, which would decrease the hydrogen abundance in the outer parts of the core and increase it at the center. The possible causes of such mixing are not clear but might again involve processes related to the solar spin-down.

Because of the inherent smoothing, the inverse analysis illustrated in Figure 3 is relatively insensitive to sharp features in the Sun. These may be detected, however, through different analysis techniques. A feature of extent smaller than the local wavelength of the mode would induce an oscillatory signal into the frequencies, depending on the phase of the eigenfunction at the location of the feature. The "frequency" and amplitude of this oscillation depend on the depth and magnitude of the feature. A sharp variation in the sound-speed gradient is in fact predicted by simple models of convective OV (e.g. Zahn 1991) with a nearly adiabatic extension of the CZ, followed by a sudden transition to the radiative gradient where the motion stops. Analyses of the observed frequencies have constrained the extent of such an OV region to be less than $0.05H_p$, and there is evidence that the transition between the CZ and the radiative region is, if anything, smoother than predicted by normal models including helium settling (e.g. Christensen-Dalsgaard et al 1995, Basu 1997). It should be kept in mind, however, that the analysis applies to a spherical average of the structure at the base of the CZ, further averaged over the time required to observe the oscillations. Penetration into the stable region could be intermittent, such that the averaging would produce a smooth profile. Furthermore, the dynamics of convection at the base of the CZ, where the estimated turnover time is comparable with the solar rotation period, are likely to be strongly coupled to rotation and hence would depend on latitude (cf Section 1.10). Hence the extent of penetration may also be latitude-dependent, which, upon averaging, would smooth the transition. Such latitude variation might be detectable, however, through separate analysis of modes of different azimuthal order *m*.

2.4 *Mode Excitation*

The inverse analyses of the oscillation frequencies are largely independent of information about the source of the oscillations: Except for the effects discussed in Section 2.5, the frequencies depend only on the structure and rotation of the solar interior. Nonetheless, it is clearly of interest to determine the causes of the oscillations.

Large-amplitude stellar pulsators have generally been found to be self-excited. Here specific regions of the star act as a heat engine: The dependence of the absorption coefficient on T and ρ is such that during pulsation, matter is heated at compression and cooled at expansion. The star becomes unstable to a given mode of pulsation if this effect dominates over damping in other parts of the star. Early calculations (e.g. Ando & Osaki 1975) indicated that similar mechanisms might operate in the Sun: The oscillations were found to be unstable in roughly the observed range of frequencies. However, significant radiative effects and, more importantly, effects of convection on the oscillations were neglected. Including them requires a time-dependent treatment of convection, taking into account the oscillations (see also Canuto 1997b). An MLT treatment of time-dependent convection (Gough 1977) was generalized to a nonlocal form by Balmforth & Gough (1990) and Balmforth (1992a), and it was recently extensively applied to solar and stellar oscillations by Houdek (1997). According to this treatment, the solar oscillations are all stable, a substantial contribution to the damping coming from the perturbation to turbulent pressure. Furthermore, the computed damping rates are in reasonable agreement with the observed lifetimes of the modes. An independent, if less direct, argument against self-excitation was presented by Kumar & Goldreich (1989), who pointed out the difficulty of accounting for the observed amplitudes when assuming self-excited oscillations.

If the modes are not self-excited, what, then, is the cause of the observed oscillations? Using earlier work by Lighthill (1952), Stein (1967) noted that the vigorously turbulent convection just beneath the solar surface is a source of acoustic noise that might excite waves in the solar atmosphere. Such noise also excites the normal modes of the Sun. Early estimates of the amplitudes (Goldreich & Keeley 1977) were somewhat pessimistic. However, later calculations have demonstrated that the observed amplitudes can be explained in terms of such stochastic excitation (e.g. Balmforth 1992b). In particular, the concentration of oscillatory power around frequencies of 3 mHz can be understood (e.g. Goldreich et al 1994): At low frequency, the modes are predominantly trapped well below the region of strong driving, whereas the highest observed frequencies are above the inverse time scales of the dominant energy-carrying convective eddies, where the driving is therefore less efficient.

Large-eddy simulations (LES) are by nature time dependent and, if fully compressible, should exhibit the excitation of acoustic waves. Indeed, in the LES, oscillations are excited spontaneously, by processes quite similar to the excitation mechanisms responsible for the solar oscillations (Stein & Nordlund 1991, Bogdan et al 1993). Although limited by the short duration of the LES in equivalent solar time, this allows, at least in principle, detailed studies of the interaction between convection and pulsations. To compare the amplitudes predicted by the simulations with those observed, the results must be scaled to correct for the small extent of the computational domain. With such scaling, the predicted amplitudes are not inconsistent with the observations.

In general, stochastic excitation of a damped oscillator leads to an average Lorenzian profile in a power spectrum of the oscillating signal (e.g. Batchelor 1956, Christensen-Dalsgaard et al 1989). However, in the solar case, the source of excitation is concentrated in a fairly narrow region near the top of the CZ; this causes an asymmetry of the line profile (e.g. Duvall et al 1993, Gabriel 1993), which has in fact been observed (e.g. Kosovichev et al 1997). The detailed shape of the profile depends on the depth of the source of excitation, hence in principle offering information about details of the turbulence spectrum beneath the solar surface. Similar information is provided by the detection of resonances at frequencies higher than those where waves are reflected by the solar atmosphere: The frequencies of these resonances also depend on the depth of excitation of the waves. An excellent fit to the observations has been achieved with a source at a depth of 140 ± 60 km beneath the photosphere (Kumar 1994); however, the sensitivity of this result to the uncertainties in the description of the physics of the waves in the near-surface layer still needs investigation.

2.5 *Properties of the Near-Surface Layer*

The scaled differences (Figure 2) between the solar frequencies and those of a standard solar model strongly indicate that the dominant errors in the model are in the superficial layers. In fact, the dependence on frequency indicates that the sources of the problems are in the solar atmosphere or in the outer roughly 10^3 km of the CZ.

What might be wrong in this region? Unfortunately, there is no unique answer. Unlike the rest of the CZ, the thermal structure of its near-surface part depends sensitively on how the convective flux is modeled. Also, this is the only region where the structure is affected by turbulent pressure, yet this was neglected in the model used in Figure 2. Potential errors in the assumed physics of the oscillations are equally likely. The oscillations are certainly not isentropic, as was assumed in the frequency calculations; nonadiabatic effects, which are significant only very near the surface, can in principle be included in the calculation, but their treatment depends critically on the assumed relation between convection and the oscillations. Convection could also have dynamical effects on the oscillations, perhaps in the form of perturbations to the turbulent pressure.

Some guidance on the likely effects may be obtained from existing calculations. Effects of the thermal structure in the strongly superadiabatic region at the top of the CZ can be illustrated by comparing the Canuto & Mazzitelli (1991) and MLT models. Scaled frequency differences between these two models are shown in Figure 4; they evidently share many of the properties of the differences shown in Figure 2 between the observations and the MLT model, including the overall magnitude (see also Paternò et al 1993, Monteiro et al 1996).

This calculation neglects dynamical effects of convection on the oscillations. Using the time-dependent MLT of Gough and Balmforth (cf Section 2.4), Houdek (1997) noted that the perturbation $\delta p_{\rm turb}$ in turbulent pressure was roughly 90° out of phase with the density perturbation. Similarly, the oscillations seen in LES appear to indicate that $\delta p_{\rm turb}$ is small. These results suggest that, for the purpose of calculating the oscillation frequencies, $\delta p_{\rm turb}$ might be

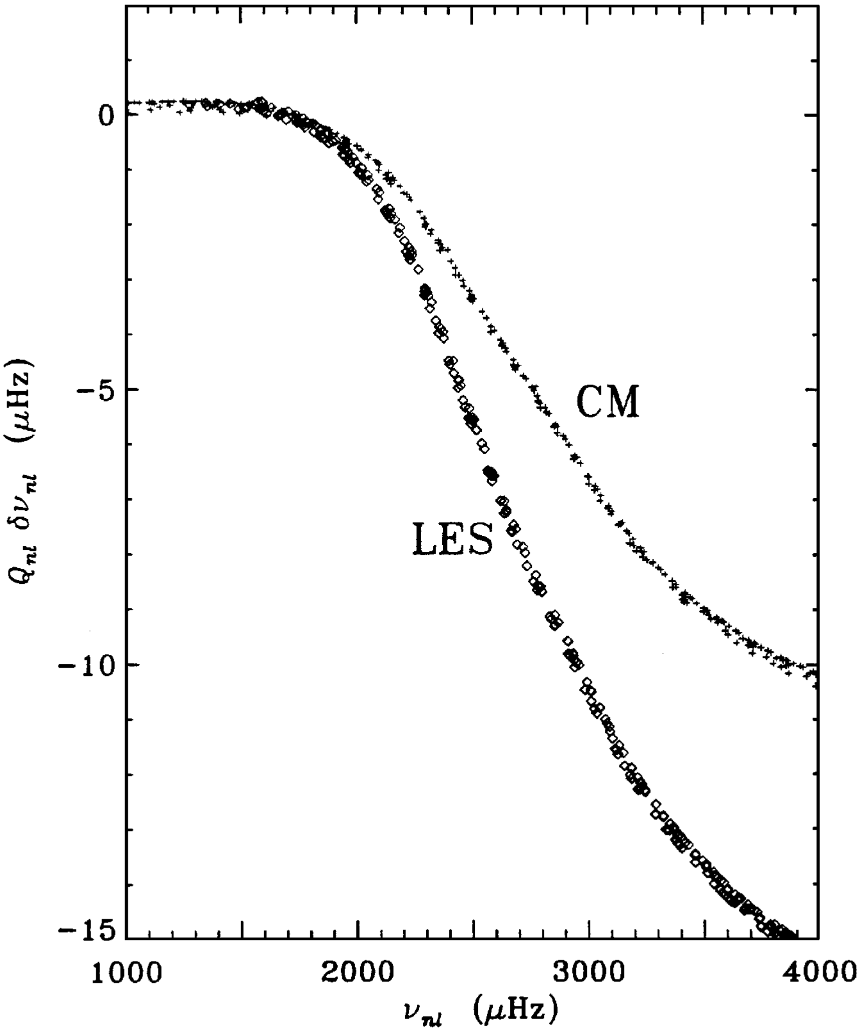

Figure 4 Scaled frequency differences between nonstandard models and standard models similar to Model S (cf caption to Figure 2), in the sense (nonstandard)–(standard). The upper points correspond to treating convection with the Canuto & Mazzitelli formulation, neglecting turbulent pressure. The lower points are based on an averaged LES, fitted continuously to an envelope model, and compared with a MLT envelope having the same depth of CZ; in the LES model, the perturbations in turbulent pressure were neglected (cf Rosenthal et al 1995).

neglected; this leads to a smaller effective compressibility Γ_1 relating density and pressure perturbations (Rosenthal et al 1995). Scaled frequency shifts from such reduction in Γ_1, estimated from a LES, are also shown in Figure 4. It is striking that these are again similar in shape and magnitude to the difference between observations and simple models. However, Antia & Basu (1997) considered the inclusion of turbulent pressure in the Canuto & Mazzitelli model; while the effect, relative to an MLT model, of the change in the thermal stratification was similar to that shown in Figure 4, the effect of $p_{\rm turb}$ was found to be very small in this case.

Inclusion of nonadiabatic effects on the frequencies with a detailed treatment of convection has so far only been attempted within the time-dependent MLT formulation. Houdek (1997) found that the nonadiabatic effects on the frequencies nearly canceled the effects of turbulent pressure discussed above, which leads to a very modest difference between the nonadiabatic frequencies and isentropic frequencies of a simple MLT model without turbulent pressure! Thus the problem remains. One might speculate that the rather larger effect of turbulent pressure in the LES (cf Figure 4), combined with the effects of nonadiabaticity, might result in better agreement with the observations.

This list of potential sources for the frequency differences is far from exhaustive. Other suggestions include effects of flows or inhomogeneities (Brown 1984, Zhugzhda & Stix 1994). An obvious and major problem is to distinguish between these possible effects and identify those that dominate. Here the f modes may play a significant role: As indicated in Section 2.2, their frequencies are largely insensitive to the mean structure of the Sun, including the reduction in the effective Γ_1. Thus the shift of their frequencies relative to the simple models would provide a measure of the importance of, for example, dynamical effects (e.g. Murawski & Roberts 1993, Ghosh et al 1995). Unfortunately, the present observations of f-mode frequencies are somewhat ambiguous (e.g. Antia 1997); however, data from the SOHO satellite are rapidly improving the situation.

2.6 *Dynamics in the CZ*

As discussed in Section 1.9, the solar surface differential rotation is believed to arise from redistribution of angular momentum within the CZ. Hydrodynamical models (e.g. Glatzmaier 1985, Gilman & Miller 1986) indicate that rotation is approximately constant on cylinder surfaces aligned with the rotation axis, as suggested by the Taylor-Proudman theorem. This simple state is not, however, confirmed by the helioseismic data. Figure 5 shows the inferred angular velocity from inversion of data from the High Altitude Observatory's LOWL instrument (Tomczyk et al 1995b, 1996). It is worth noting that no information about the surface rotation rate was used in the analysis; nonetheless, the inferred

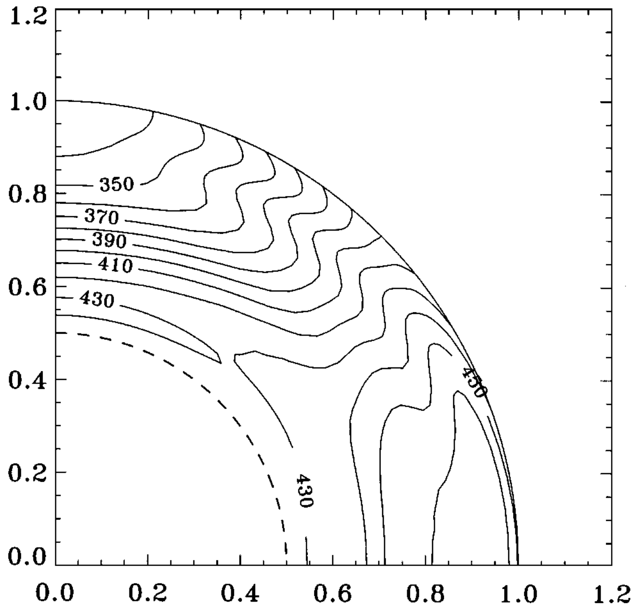

Figure 5 Angular velocity $\Omega(r,\theta)/2\pi$ in the solar interior, inferred from inversion of frequency splittings obtained with the LOWL instrument (Tomczyk et al 1995b, 1996). Rotation is shown in one quadrant of a cross section of the Sun, with the rotation axis pointing upwards. Selected contours are labeled with $\Omega(r,\theta)/2\pi$, in nanohertz. Inside the dashed circle, and very near the axis, inversion is not possible with these data.

angular velocity near the surface is in fact close to the measured surface value. Rotation evidently depends mainly on latitude within the CZ, although with a slight increase with depth near the surface. Near the base of the CZ there is a transition, in the tachocline (Spiegel & Zahn 1992), to nearly uniform rotation in the radiative interior. The width and precise location of this tachocline are of great interest but so far somewhat uncertain. Much of the apparent extent of the transition in Figure 5 is due to the finite resolution of the inversion; Kosovichev (1996a) found that the width was approximately $0.05R$, with the transition centered slightly below the base of the CZ, whereas analysis of more recent data (Basu 1997, Charbonneau et al 1997) found only an upper limit on the width. Also, the transition appears to extend over a larger part of the CZ at high latitudes.

No current models account for this behavior. The failure of the models by Gilman and Glatzmaier suggests that anisotropic transport of angular momentum is required. Results such as those shown in Figure 5 obviously will be instrumental in testing the models of turbulent transport presented in Section 1.9.

Attempts have been made to model the rotation of the radiative interior, on the assumption of a substantial loss of angular momentum in the solar wind over solar evolution; transport between the radiative interior and the CZ was achieved through turbulent transport resulting from rotationally induced instabilities (Pinsonneault et al 1989). These models generally predict an increase in angular velocity with depth, with the deep interior rotating substantially more rapidly than the surface. As is evident from Figure 5, this behavior is inconsistent with the helioseismically inferred angular velocity. Thus, other mechanisms appear to be responsible for the transport of angular momentum.

2.7 *Where Do We Go from Here?*

Helioseismic data allow us to test the detailed predictions of models of turbulence as applied to convection and other dynamical phenomena of the solar interior. Although the results are not yet conclusive, improved treatments of the near-surface region through more realistic descriptions of convection, including turbulent pressure, seem required to match the observed frequencies. Also, the excitation of the observed oscillations is providing information about the depth distribution of convection, and the inferred angular velocity places strict constraints on models of angular-momentum transport in and below the solar CZ.

Data on solar oscillations are expanding rapidly, through the Global Oscillation Network Group (GONG) network (Harvey et al 1996) and the helioseismic instruments on the SOHO satellite (e.g. Scherrer et al 1996). High-resolution data from the SOI/MDI project on SOHO are providing very detailed information about the solar near-surface region. In particular, the so-called time-distance helioseismology has provided evidence for subsurface flows and temperature perturbations associated with supergranular convection (e.g. Kosovichev 1996b), and there is hope that such local-area techniques may uncover the elusive giant cells of solar convection.

Data on oscillations of distant stars will also be of great importance, despite the unavoidable limitations in the observations, by expanding the range of conditions that can be probed. An early example of the importance of convection to the study of pulsating stars is the return to stability at the cool side of the Cepheid instability strip, which appears to be due to the onset of convection (e.g. Baker & Gough 1979). Also, one would expect stochastic excitation of solar-like oscillations in stars with vigorous surface convection. Houdek (1997) made an extensive survey of the stability properties and excitation of oscillations in main-sequence stars, using nonlocal time-dependent MLT. The results are summarized by Figure 6. At effective temperatures T_{eff} greater than

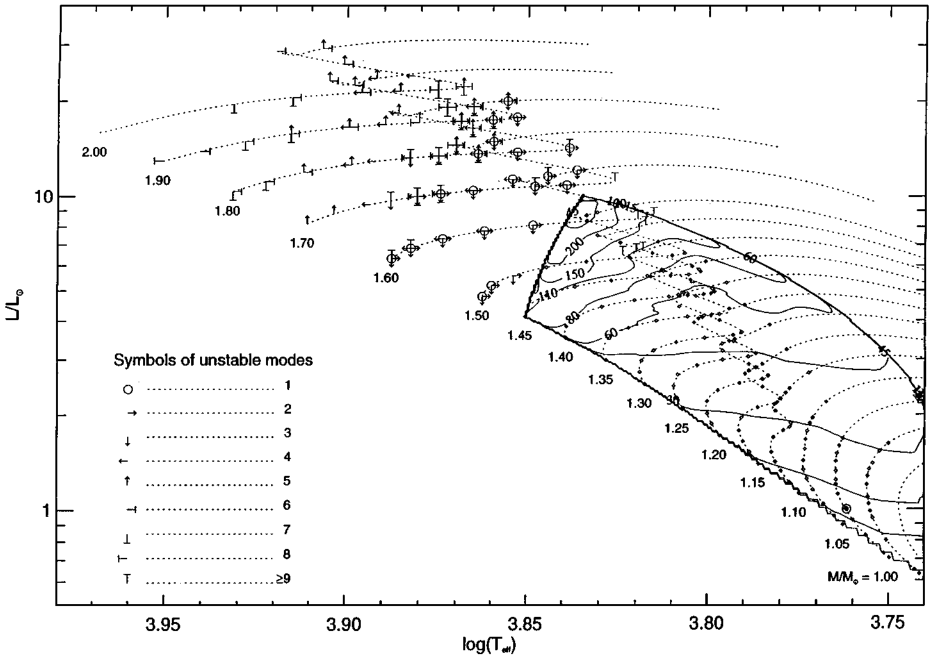

Figure 6 Unstable modes and mean velocities of stochastically excited oscillations, computed with the nonlocal MLT of Balmforth & Gough (1990). The positions of the stars are indicated by their effective temperature $T_{\rm eff}$ and luminosity in units of the solar luminosity $L_\odot$. The dashed lines show evolution tracks, labeled by the stellar mass in solar units. The unstable modes are shown with symbols according to their radial order, as indicated at the lower left. The contours show amplitudes of oscillations, excited by the turbulent convection; selected contours have been labeled by the amplitude, in centimeters per second. (From Houdek 1997.)

$\log T_{\rm eff} \simeq 3.85$, several modes are unstable; this corresponds to the extension to the main sequence of the Cepheid instability strip and contains the so-called δ Scuti stars. At lower effective temperature, the instability is suppressed by convection, and the contours show the predicted amplitudes of stochastically excited modes; these are substantially higher for higher-mass stars than for the Sun.

Even at these higher amplitudes, the observational problems in detecting the oscillations are formidable. Nonetheless, many attempts have been made, of which the most plausible detection is probably for the star η Bootis (Kjeldsen et al 1995; see, however, Brown et al 1997). Successful observations of such oscillations would enable the use of the frequencies to probe the interior structure and possibly rotation of a range of stars, and, equally important for the understanding of stellar turbulence, would provide information about the excitation and hence the properties of convection in other stars. Indeed it is plausible that the oscillations observed in cool giants such as Arcturus (e.g. Innis et al 1988, Belmonte et al 1990) or giants in the cluster 47 Tucanae (Edmonds & Gilliland 1996) are excited by the same mechanism as in the case of the solar oscillations.

Given the current successes and the ongoing strong efforts, we have high hopes that we shall be able to use the Sun and other oscillating stars as laboratories for the study of turbulence under astrophysical conditions.

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